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THE  
MATHEMATICAL PRINCIPLES  
OF  
NATURAL PHILOSOPHY.  
BY  
SIR ISAAC NEWTON.

Translated into English  
BY ANDREW MOTTE.

TO WHICH ARE ADDED,

*Newton's System of the World;*  
A SHORT  
Comment on, and Defence of, the Principia,  
BY W. EMERSON.

WITH  
THE LAWS OF THE MOON'S MOTION  
According to Gravity.  
BY JOHN MACHIN,  
Astron., Prof. at Gresh., and Sec. to the Roy. Soc.

A new Edition,

(With the LIFE of the AUTHOR ; and a PORTRAIT, taken from the Bust in  
the Royal Observatory at Greenwich)

CAREFULLY REVISED AND CORRECTED BY  
W. DAVIS,

Author of the "Treatise on Land Surveying," the "Use of the Globes,"  
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*of*  
THE SYSTEM OF THE WORLD.

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## SYSTEM OF THE WORLD.



IT was the antient opinion of not a few, in the earliest ages of philosophy, that the fixed stars stood immovable in the highest parts of the world ; that under the fixed stars the planets were carried about the sun ; that the earth, as one of the planets, described an annual course about the sun, while by a diurnal motion it was in the mean time revolved about its own axis ; and that the sun, as the common fire which served to warm the whole, was fixed in the centre of the universe.

This was the philosophy taught of old by *Philolaus*, *Aristarchus* of Samos, *Plato* in his riper years, and the whole sect of the *Pythagoreans* ; and this was the judgment of *Anaximander*, more antient than any of them ; and of that wise king of the Romans, *Numa Pompilius*, who, as a symbol of the figure of the world with the sun in the centre, erected a temple in honour of *Vesta*, of a round form, and ordained perpetual fire to be kept in the middle of it.

The *Egyptians* were early observers of the heavens ; and from them, probably, this philosophy was spread abroad among other nations ; for from them it was, and the nations about them, that the *Greeks*, a people of themselves more addicted to the study of philology than of nature, derived their first, as well as foundest, notions of philosophy ; and in the vestal ceremonies we may yet trace the antient spirit of the *Egyptians* ; for it was their way to deliver their mysteries, that is, their philosophy of things above the vulgar way of thinking, under the veil of religious rites and hieroglyphic symbols.

It is not to be denied but that *Anaxagoras*, *Democritus*, and others, did now and then start up, who would have it that the earth possessed the centre of the world, and that the

stars of all sorts were revolved towards the west about the earth quiescent in the centre, some at a swifter, others at a slower rate.

However, it was agreed on both sides that the motions of the celestial bodies were performed in spaces altogether free and void of resistance. The whim of solid orbs was of a later date, introduced by *Eudoxus*, *Calippus*, and *Aristotle*; when the antient philosophy began to decline, and to give place to the new prevailing fictions of the Greeks.

But, above all things, the phænomena of comets can by no means consist with the notion of solid orbs. The *Chaldeans*, the most learned astronomers of their time, looked upon the comets (which of antient times before had been numbered among the celestial bodies) as a particular sort of planets, which, describing very eccentric orbits, presented themselves to our view only by turns, viz. once in a revolution, when they descended into the lower parts of their orbits.

And as it was the unavoidable consequence of the hypothesis of solid orbs, while it prevailed, that the comets should be thrust down below the moon, so no sooner had the late observations of astronomers restored the comets to their antient places in the higher heavens, but these celestial spaces were at once cleared of the incumbrance of solid orbs, which by these observations were broke into pieces, and discarded for ever.

Whence it was that the planets came to be retained within any certain bounds in these free spaces, and to be drawn off from the rectilinear courses, which, left to themselves, they should have pursued, into regular revolutions in curvilinear orbits, are questions which we do not know how the antients explained; and probably it was to give some sort of satisfaction to this difficulty that solid orbs were introduced.

The later philosophers pretend to account for it either by the action of certain vortices, as *Kepler* and *Des Cartes*; or by some other principle of impulse or attraction, as *Borelli*, *Hooke*, and others of our nation; for, from the laws of motion,

it is most certain that these effects must proceed from the action of some force or other.

But our purpose is only to trace out the quantity and properties of this force from the phænomena (p. 174, vol. 1 \*), and to apply what we discover in some simple cases as principles, by which, in a mathematical way, we may estimate the effects thereof in more involved cases; for it would be endless and impossible to bring every particular to direct and immediate observation.

We said, *in a mathematical way*, to avoid all questions about the nature or quality of this force, which we would not be understood to determine by any hypothesis; and therefore call it by the general name of a centripetal force, as it is a force which is directed towards some centre; and as it regards more particularly a body in that centre, we call it circum-solar, circum-terrestrial, circum-jovial; and in like manner in respect of other central bodies.

That by means of centripetal forces the planets may be retained in certain orbits, we may easily understand, if we consider the motions of projectiles (p. 2, 3, 4, vol. 1); for a stone projected is by the pressure of its own weight forced out of the rectilinear path, which by the projection alone it should have pursued, and made to describe a curve line in the air; and through that crooked way is at last brought down to the ground; and the greater the velocity is with which it is projected, the farther it goes before it falls to the earth. We may therefore suppose the velocity to be so increased, that it would describe an arc of 1, 2, 5, 10, 100, 1000 miles before it arrived at the earth, till at last, exceeding the limits of the earth, it should pass quite by without touching it.

Let AFB represent the surface of the earth, C its centre, VD, VE, VF, the curve lines which a body would describe, if projected in an horizontal direction from the top of an high mountain successively with more and more velocity (vide vol. 2, p. 180); and, because the celestial motions are scarcely retarded by the little or no resistance of the spaces in which they are performed, to keep up the parity of cases, let us suppose either that there is no air about the earth, or at least that

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\* The References are to the New Edition of the Mathematical Principles, &c.

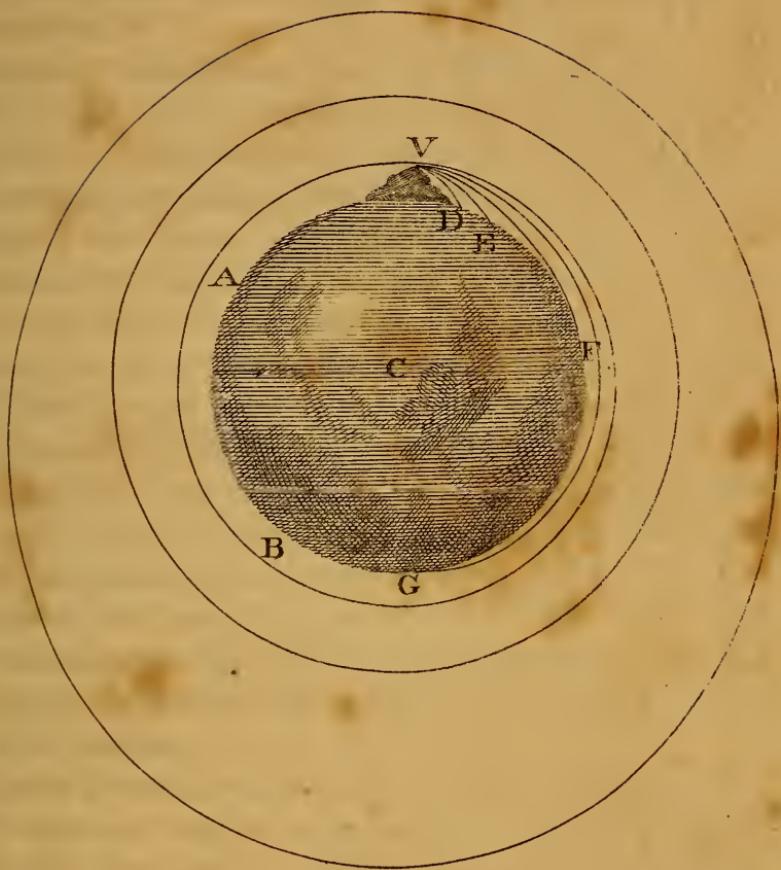
it is endowed with little or no power of resisting ; and for the same reason that the body projected with a less velocity describes the lesser arc VD, and with a greater velocity the greater arc VE, and, augmenting the velocity, it goes farther and farther to F and G, if the velocity was still more and more augmented, it would reach at last quite beyond the circumference of the earth, and return to the mountain from which it was projected.

And since the areas which by this motion it describes by a radius drawn to the centre of the earth are (by prop. 1, book 1, *Princip. Math.*) proportional to the times in which they are described, its velocity, when it returns to the mountain, will be no less than it was at first ; and, retaining the same velocity, it will describe the same curve over and over, by the same law.

But if we now imagine bodies to be projected in the directions of lines parallel to the horizon from greater heights, as of 5, 10, 100, 1000, or more miles, or rather as many semi-diameters of the earth, those bodies, according to their different velocity, and the different force of gravity in different heights, will describe arcs either concentric with the earth, or variously eccentric, and go on revolving through the heavens in those trajectories, just as the planets do in their orbs.

As when a stone is projected obliquely, that is, any way but in the perpendicular direction, the perpetual deflection thereof towards the earth from the right line in which it was projected is a proof of its gravitation to the earth, no less certain than its direct descent when only suffered to fall freely from rest ; so the deviation of bodies moving in free spaces from rectilinear paths, and perpetual deflection therefrom towards any place, is a sure indication of the existence of some force which from all quarters impels those bodies towards that place.

And as, from the supposed existence of gravity, it necessarily follows that all bodies about the earth must press downwards, and therefore must either descend directly to the earth, if they are let fall from rest, or at least perpetually deviate from right lines towards the earth, if they are projected ob-





liuely ; so from the supposed existence of a force directed to any centre, it will follow, by the like necessity, that all bodies upon which this force acts must either descend directly to that centre, or at least deviate perpetually towards it from right lines, if otherwise they should have moved obliquely in these right lines.

And how from the motions given we may infer the forces, or from the forces given we may determine the motions, is shewn in the two first books of our *Principles of Philosophy*.

If the earth is supposed to stand still, and the fixed stars to be revolved in free spaces in the space of 24 hours, it is certain the forces by which the fixed stars are retained in their orbs are not directed to the earth, but to the centres of the several orbs, that is, of the several parallel circles, which the fixed stars, declining to one side and the other from the equator, describe daily ; also that by radii drawn to the centres of those orbs the fixed stars describe areas exactly proportional to the times of description. Then, because the periodic times are equal (by cor. 3, prop. 4, book 1), it follows that the centripetal forces are as the radii of the several orbs, and that they will perpetually revolve in the same orbs. And the like consequences may be drawn from the supposed diurnal motion of the planets.

That forces should be directed to no body on which they physically depend, but to innumerable imaginary points in the axis of the earth, is an hypothesis too incongruous. It is more incongruous still that those forces should increase exactly in proportion of the distances from this axis ; for this is an indication of an increase to immensity, or rather to infinity ; whereas the forces of natural things commonly decrease in receding from the fountain from which they flow. But, what is yet more absurd, neither are the areas described by the same star proportional to the times, nor are its revolutions performed in the same orb ; for as the star recedes from the neighbouring pole, both areas and orb increase ; and from the increase of the area it is demonstrated that the forces are not directed to the axis of the earth. And this difficulty (cor. 1, prop. 2) arises from the twofold motion

that is observed in the fixed stars, one diurnal round the axis of the earth, the other exceedingly slow round the axis of the ecliptic. And the explication thereof requires a composition of forces so perplexed and so variable, that it is hardly to be reconciled with any physical theory.

That there are centripetal forces actually directed to the bodies of the sun, of the earth, and other planets, I thus infer.

The moon revolves about our earth, and by radii drawn to its centre (p. 167) describes areas nearly proportional to the times in which they are described, as is evident from its velocity compared with its apparent diameter; for its motion is slower when its diameter is less (and therefore its distance greater), and its motion is swifter when its diameter is greater.

The revolutions of the satellites of Jupiter about that planet are more regular (p. 162); for they describe circles concentric with Jupiter by equable motions, as exactly as our senses can distinguish.

And so the satellites of Saturn are revolved about this planet with motions nearly (p. 164) circular and equable, scarcely disturbed by any eccentricity hitherto observed.

That Venus and Mercury are revolved about the sun, is demonstrable from their moon-like appearances (p. 165): when they shine with a full face, they are in those parts of their orbs which in respect of the earth lie beyond the sun; when they appear half full, they are in those parts which lie over against the sun; when horned, in those parts which lie between the earth and the sun; and sometimes they pass over the sun's disk, when directly interposed between the earth and the sun.

And Venus, with a motion almost uniform, describes an orb nearly circular and concentric with the sun.

But Mercury, with a more eccentric motion, makes remarkable approaches to the sun, and goes off again by turns; but it is always swifter as it is near to the sun, and therefore by a radius drawn to the sun still describes areas proportional to the times.

Lastly, that the earth describes about the sun, or the sun about the earth, by a radius from the one to the other, areas exactly proportional to the times, is demonstrable from the apparent diameter of the sun compared with its apparent motion.

These are astronomical experiments; from which it follows, by prop. 1, 2, 3, in the first book of our *Principles*, and their corollaries (p. 167, 168, 171), that there are centripetal forces actually directed (either accurately or without considerable error) to the centres of the earth, of Jupiter, of Saturn, and of the sun. In Mercury, Venus, Mars, and the lesser planets, where experiments are wanting, the arguments from analogy must be allowed in their place.

That those forces (p. 167, 168, 171) decrease in the duplicate proportion of the distances from the centre of every planet, appears by cor. 6, prop. 4, book 1; for the periodic times of the satellites of Jupiter are one to another (p. 162, 163) in the sesquiplicate proportion of their distances from the centre of this planet.

This proportion has been long ago observed in those satellites; and Mr. *Flamsted*, who had often measured their distances from Jupiter by the micrometer, and by the eclipses of the satellites, wrote to me, that it holds to all the accuracy that possibly can be discerned by our senses. And he sent me the dimensions of their orbits taken by the micrometer, and reduced to the mean distance of Jupiter from the earth, or from the sun, together with the times of their revolutions, as follows:

The greatest elongation of the satellites from the centre of Jupiter as seen from the sun.	The periodic times of their revolutions.
	d. h. m. s.
1st...1 48 or 108	1 18 28 36
2d...3 01 or 181	3 13 17 54
3d...4 46 or 286	7 03 59 36
4th...8 13½ or 493½	16 18 5 13

Whence the sesquiplicate proportion may be easily seen. For example; the  $16^d. 18^h. 05' 13''$  is to the time  $1^d. 18^h. 28' 36''$  as  $493\frac{1}{2}'' \times \sqrt{493\frac{1}{2}''}$  to  $108'' \times \sqrt{108''}$ , neglecting those small fractions which, in observing, cannot be certainly determined.

Before the invention of the micrometer, the same distances were determined in semi-diameters of Jupiter thus :

	Distance of the	1st.	2d.	3d.	4th.
By Galileo .....	6	10	16	28	
Simon Marius....	6	10	16	26	
Cassini.....	5	8	13	23	
Borelli more exactly } ..	$5\frac{2}{3}$	$8\frac{2}{3}$	14	$24\frac{2}{3}$	

After the invention of the micrometer.

By Townley .....	5,51	8,78	13,47	24,72
Flamsted .....	5,31	8,85	13,98	24,23
More accurately by the eclipses }	5,578	8,876	14,159	24,903

And the periodic times of those satellites, by the observations of Mr. Flamsted, are  $1^d. 18^h. 28' 36''$  |  $3^d. 17^h. 17' 54''$  |  $7^d. 3^h. 59' 36''$  |  $16^d. 18^h. 5' 13''$ , as above.

And the distances thence computed are 5,578 | 8,876 | 14,168 | 24,968, accurately agreeing with the distances by observation.

Cassini assures us (p. 164, 165) that the same proportion is observed in the circum-saturnal planets. But a longer course of observations is required before we can have a certain and accurate theory of those planets.

In the circum-solar planets, Mercury and Venus, the same proportion holds with great accuracy, according to the dimensions of their orbs, as determined by the observations of the best astronomers.

That Mars is revolved about the sun is demonstrated from the phases which it shews, and the proportion of its apparent diameters (p. 165, 166, and 167); for from its appearing full near conjunction with the sun, and gibbous in its quadratures, it is certain that it surrounds the sun.

And since its diameter appears about five times greater when in opposition to the sun than when in conjunction therewith, and its distance from the earth is reciprocally as its apparent diameter, that distance will be about five times less when in opposition to than when in conjunction with the sun ; but in both cases its distance from the sun will be nearly about the same with the distance which is inferred from its gibbous appearance in the quadratures. And as it encompasses the sun at almost equal distances, but in respect of the earth is very unequally distant, so by radii drawn to the sun it describes areas nearly uniform ; but by radii drawn to the earth, it is sometimes swift, sometimes stationary, and sometimes retrograde.

That Jupiter, in a higher orb than Mars, is likewise revolved about the sun, with a motion nearly equable, as well in distance as in the areas described, I infer thus.

Mr. *Flamsted* assured me, by letters, that all the eclipses of the innermost satellite which hitherto have been well observed do agree with his theory so nearly, as never to differ therefrom by two minutes of time ; that in the outmost the error is little greater ; in the outmost but one, scarcely three times greater ; that in the innermost but one the difference is indeed much greater, yet so as to agree as nearly with his computations as the moon does with the common tables ; and that he computes those eclipses only from the mean motions corrected by the equation of light discovered and introduced by Mr. *Romer*. Supposing, then, that the theory differs by a less error than that of  $2'$  from the motion of the outmost satellite as hitherto described, and taking as the periodic time  $16^d. 18^h. 5' 13''$  to  $2'$  in time, so is the whole circle or  $360^\circ$  to the arc  $1' 48''$ , the error of Mr. *Flamsted*'s computation, reduced to the satellite's orbit, will be less than  $1' 48''$ ; that is, the longitude of the satellite, as seen from the centre of Jupiter, will be determined with a less error than  $1' 48''$ . But when the satellite is in the middle of the shadow, that longitude is the same with the heliocentric longitude of Jupiter ; and, therefore, the hypothesis which Mr. *Flamsted* follows, viz. the *Copernican*, as improved by *Kepler*, and (as to the motion of

Jupiter) lately corrected by himself, rightly represents that longitude within a less error than  $1' 48''$ ; but by this longitude, together with the geocentric longitude, which is always easily found, the distance of Jupiter from the sun is determined; which must, therefore, be the very same with that which the hypothesis exhibits. For that greatest error of  $1' 48''$  that can happen in the heliocentric longitude is almost insensible, and quite to be neglected, and perhaps may arise from some yet undiscovered eccentricity of the satellite; but since both longitude and distance are rightly determined, it follows of necessity that Jupiter, by radii drawn to the sun, describes areas so conditioned as the hypothesis requires, that is, proportional to the times.

And the same thing may be concluded of Saturn from his satellite, by the observations of Mr. *Huygens* and Dr. *Halley*; though a longer series of observations is yet wanting to confirm the thing, and to bring it under a sufficiently exact computation.

For if Jupiter was viewed from the sun, it would never appear retrograde nor stationary, as it is seen sometimes from the earth, but always to go forward with a motion nearly uniform (p. 166). And from the very great inequality of its apparent geocentric motion, we infer (by prop. 3, cor. 4) that the force by which Jupiter is turned out of a rectilinear course, and made to revolve in an orb, is not directed to the centre of the earth. And the same argument holds good in Mars and in Saturn. Another centre of these forces is therefore to be looked for (by prop. 2 and 3, and the corollaries of the latter), about which the areas described by radii intervening may be equable; and that this is the sun, we have proved already in Mars and Saturn nearly; but accurately enough in Jupiter. It may be alledged that the sun and planets are impelled by some other force equally and in the direction of parallel lines; but by such a force (by cor. 6 of the laws of motion) no change would happen in the situation of the planets one to another, nor any sensible effect follow: but our busines is with the causes of sensible effects. Let us, therefore, neglect every such force as imaginary and precarious, and of no use

in the phænomena of the heavens ; and the whole remaining force by which Jupiter is impelled will be directed (by prop. 3, cor. 1) to the centre of the sun.

The distances of the planets from the sun come out the same, whether, with *Tycho*, we place the earth in the centre of the system, or the sun with *Copernicus* : and we have already proved that these distances are true in Jupiter.

*Kepler* and *Bullialdus* have, with great care (p. 165), determined the distances of the planets from the sun ; and hence it is that their tables agree best with the heavens. And in all the planets, in Jupiter and Mars, in Saturn and the Earth, as well as in Venus and Mercury, the cubes of their distances are as the squares of their periodic times ; and therefore (by cor. 6, prop. 4) the centripetal circum-solar force throughout all the planetary regions decreases in the duplicate proportion of the distances from the sun. In examining this proportion, we are to use the mean distances, or the transverse semi-axes of the orbits (by prop. 15), and to neglect those little fractions, which, in defining the orbits, may have arisen from the insensible errors of observation, or may be ascribed to other causes which we shall afterwards explain. And thus we shall always find the said proportion to hold exactly ; for the distances of Saturn, Jupiter, Mars, the Earth, Venus, and Mercury, from the sun, drawn from the observations of astronomers, are, according to the computation of *Kepler*, as the numbers 951000, 519650, 152350, 100000, 72400, 38806 ; by the computation of *Bullialdus*, as the numbers 954198, 522520, 152350, 100000, 72398, 38585 ; and from the periodic times they come out 953806, 520116, 152399, 100000, 72333, 38710. Their distances, according to *Kepler* and *Bullialdus*, scarcely differ by any sensible quantity, and where they differ most the distances drawn from the periodic times fall in between them.

That the circum-terrestrial force likewise decreases in the duplicate proportion of the distances, I infer thus.

The mean distance of the moon from the centre of the earth, is, in semi-diameters of the earth, according to *Ptolemy*, *Kepler* in his *Ephemerides*, *Bullialdus*, *Hevelius*, and

*Ricciolus*, 59; according to *Flamsted*,  $59\frac{4}{3}$ ; according to *Tycho*,  $56\frac{1}{2}$ ; to *Vendelin*, 60; to *Copernicus*,  $60\frac{1}{3}$ ; to *Kircher*,  $62\frac{1}{2}$  (p. 169, 170, 171).

But *Tycho*, and all that follow his tables of refraction, making the refractions of the sun and moon (altogether against the nature of light) to exceed those of the fixed stars, and that by about four or five minutes in the horizon, did thereby augment the horizontal parallax of the moon by about the like number of minutes; that is, by about the 12th or 15th part of the whole parallax. Correct this error, and the distance will become 60 or 61 semi-diameters of the earth, nearly agreeing with what others have determined.

Let us, then, assume the mean distance of the moon 60 semi-diameters of the earth, and its periodic time in respect of the fixed stars  $27^d. 7^h. 48'$ , as astronomers have determined it. And (by cor. 6, prop. 4) a body revolved in our air, near the surface of the earth supposed at rest, by means of a centripetal force which should be to the same force at the distance of the moon in the reciprocal duplicate proportion of the distances from the centre of the earth, that is, as 3600 to 1, would (excluding the resistance of the air) complete a revolution in  $1^h. 24' 27''$ .

Suppose the circumference of the earth to be 123249600 *Paris* feet, as has been determined by the late mensuration of the *French* (vide p. 188); then the same body, deprived of its circular motion, and falling by the impulse of the same centripetal force as before, would, in one second of time, describe  $15\frac{1}{2}$  *Paris* feet.

This we infer by a calculus formed upon prop. 36, and it agrees with what we observe in all bodies about the earth. For by the experiments of pendulums, and a computation raised thereon, Mr. *Huygens* has demonstrated that bodies falling by all that centripetal force with which (of whatever nature it is) they are impelled near the surface of the earth, do, in one second of time, describe  $15\frac{1}{2}$  *Paris* feet.

But if the earth is supposed to move, the earth and moon together (by cor. 4 of the laws of motion, and prop. 57) will be revolved about their common centre of gravity. And the

moon (by prop. 60) will in the same periodic time,  $27^{\text{d}}. 7^{\text{h}}. 43'$ , with the same circum-terrestrial force diminished in the duplicate proportion of the distance, describe an orbit whose semi-diameter is to the semi-diameter of the former orbit, that is, to 60 semi-diameters of the earth, as the sum of both the bodies of the earth and moon to the first of two mean proportionals between this sum and the body of the earth; that is, if we suppose the moon (on account of its mean apparent diameter  $31\frac{1}{2}'$ ) to be about  $\frac{1}{42}$  of the earth, as 43 to

$\sqrt{42 + 43^2}$ , or as about 128 to 127. And therefore the semi-diameter of the orbit, that is, the distance between the centres of the moon and earth, will in this case be  $60\frac{1}{2}$  semi-diameters of the earth, almost the same with that assigned by *Copernicus*, which the *Tychonic* observations by no means disprove; and, therefore, the duplicate proportion of the decrement of the force holds good in this distance. I have neglected the increment of the orbit which arises from the action of the sun as inconsiderable; but if that is subducted, the true distance will remain about  $60\frac{4}{5}$  semi-diameters of the earth.

But farther (p. 167); this proportion of the decrement of the forces is confirmed from the eccentricity of the planets, and the very slow motion of their apses; for (by the corollaries of prop. 45) in no other proportion could the circum-solar planets once in every revolution descend to their least and once ascend to their greatest distance from the sun, and the places of those distances remain immovable. A small error from the duplicate proportion would produce a motion of the apses considerable in every revolution, but in many enormous.

But now, after innumerable revolutions, hardly any such motion has been perceived in the orbs of the circum-solar planets. Some astronomers affirm that there is no such motion; others reckon it no greater than what may easily arise from the causes hereafter to be assigned, and is of no moment in the present question.

We may even neglect the motion of the moon's apsis (p. 167, 168), which is far greater than in the circum-solar

planets, amounting in every revolution to three degrees ; and from this motion it is demonstrable that the circum-terrestrial force decreases in no less than the duplicate, but far less than the triplicate proportion of the distance ; for if the duplicate proportion was gradually changed into the triplicate, the motion of the apsis would thereby increase to infinity ; and, therefore, by a very small mutation, would exceed the motion of the moon's apsis. This slow motion arises from the action of the circum-solar force, as we shall afterwards explain. But, secluding this cause, the apsis or apogee of the moon will be fixed, and the duplicate proportion of the decrease of the circum-terrestrial force in different distances from the earth will accurately take place.

Now that this proportion has been established, we may compare the forces of the several planets among themselves (p. 178).

In the mean distance of Jupiter from the earth, the greatest elongation of the outmost satellite from Jupiter's centre (by the observations of Mr. *Flamsted*) is  $8' 13''$ ; and therefore the distance of the satellite from the centre of Jupiter is to the mean distance of Jupiter from the centre of the sun as 124 to 52012, but to the mean distance of Venus from the centre of the sun as 124 to 7234; and their periodic times are  $16\frac{3}{4}^d.$  and  $224\frac{3}{2}^d.$ ; and from hence (according to cor. 2, prop. 4), dividing the distances by the squares of the times, we infer that the force by which the satellite is impelled towards Jupiter is to the force by which Venus is impelled towards the sun as 442 to 143; and if we diminish the force by which the satellite is impelled in the duplicate proportion of the distance 124 to 7234, we shall have the circum-jovial force in the distance of Venus from the sun to the circum-solar force by which Venus is impelled as  $\frac{1}{160}$  to 143, or as 1 to 1100; wherefore at equal distances the circum-solar force is 1100 times greater than the circum-jovial.

And, by the like computation, from the periodic time of the satellite of Saturn  $15^d. 22^h.$  and its greatest elongation from Saturn, while that planet is in its mean distance from us,  $3' 20''$ , it follows that the distance of this satellite from Saturn's

centre is to the distance of Venus from the sun as  $92\frac{2}{3}$  to 7234; and from thence that the absolute circum-solar force is 2360 times greater than the absolute circum-saturnal.

From the regularity of the heliocentric and irregularity of the geocentric motions of Venus, of Jupiter, and the other planets, it is evident (by cor. 4, prop. 3) that the circum-terrestrial force, compared with the circum-solar, is very small.

*Ricciolus* and *Vendelin* have severally tried to determine the sun's parallax from the moon's dichotomies observed by the telescope, and they agree that it does not exceed half a minute.

*Kepler*, from *Tycho*'s observations and his own, found the parallax of Mars insensible, even in opposition to the sun, when that parallax is something greater than the sun's.

*Flamsted* attempted the same parallax with the micrometer in the perigeon position of Mars, but never found it above  $25''$ ; and thence concluded the sun's parallax at most  $10''$ .

Whence it follows that the distance of the moon from the earth bears no greater proportion to the distance of the earth from the sun than 29 to 10000; nor to the distance of Venus from the sun than 29 to 7233.

From which distances, together with the periodic times, by the method above explained, it is easy to infer that the absolute circum-solar force is greater than the absolute circum-terrestrial force at least 229400 times.

And though we were only certain, from the observations of *Ricciolus* and *Vendelin*, that the sun's parallax was less than half a minute, yet from this it will follow that the absolute circum-solar force exceeds the absolute circum-terrestrial force 8500 times.

By the like computations I happened to discover an analogy that is observed between the forces and the bodies of the planets; but, before I explain this analogy, the apparent diameters of the planets in their mean distances from the earth must be first determined.

Mr. *Flamsted* (p. 163), by the micrometer, measured the diameter of Jupiter  $40''$  or  $41''$ ; the diameter of Saturn's ring  $50''$ ; and the diameter of the sun about  $32' 13''$  (p. 165).

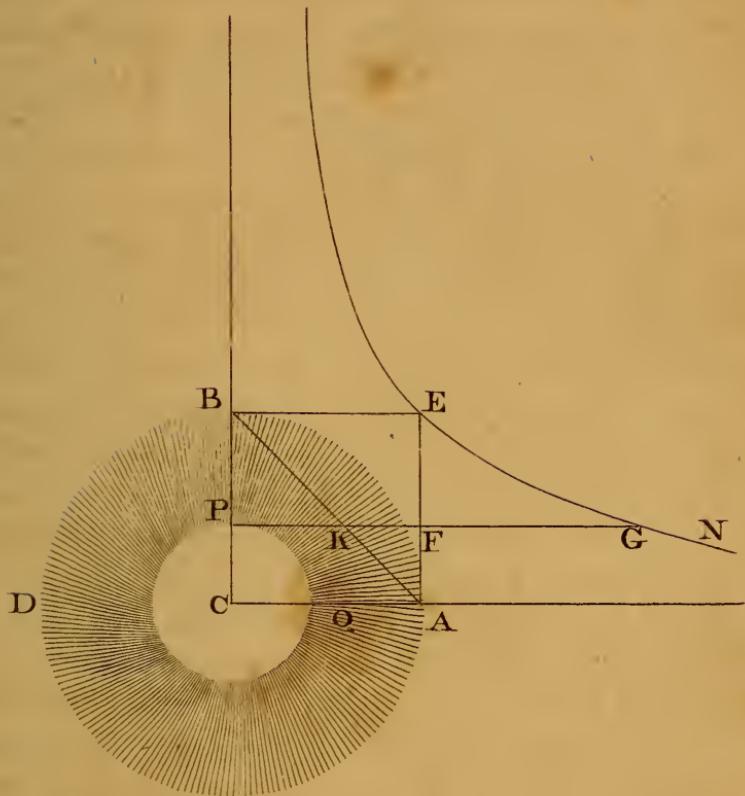
But the diameter of Saturn is to the diameter of the ring, according to Mr. *Huygens* and Dr. *Halley*, as 4 to 9; according to *Galletius*, as 4 to 10; and according to *Hooke* (by a telescope of 60 feet), as 5 to 12. And from the mean proportion, 5 to 12, the diameter of Saturn's body is inferred about 21".

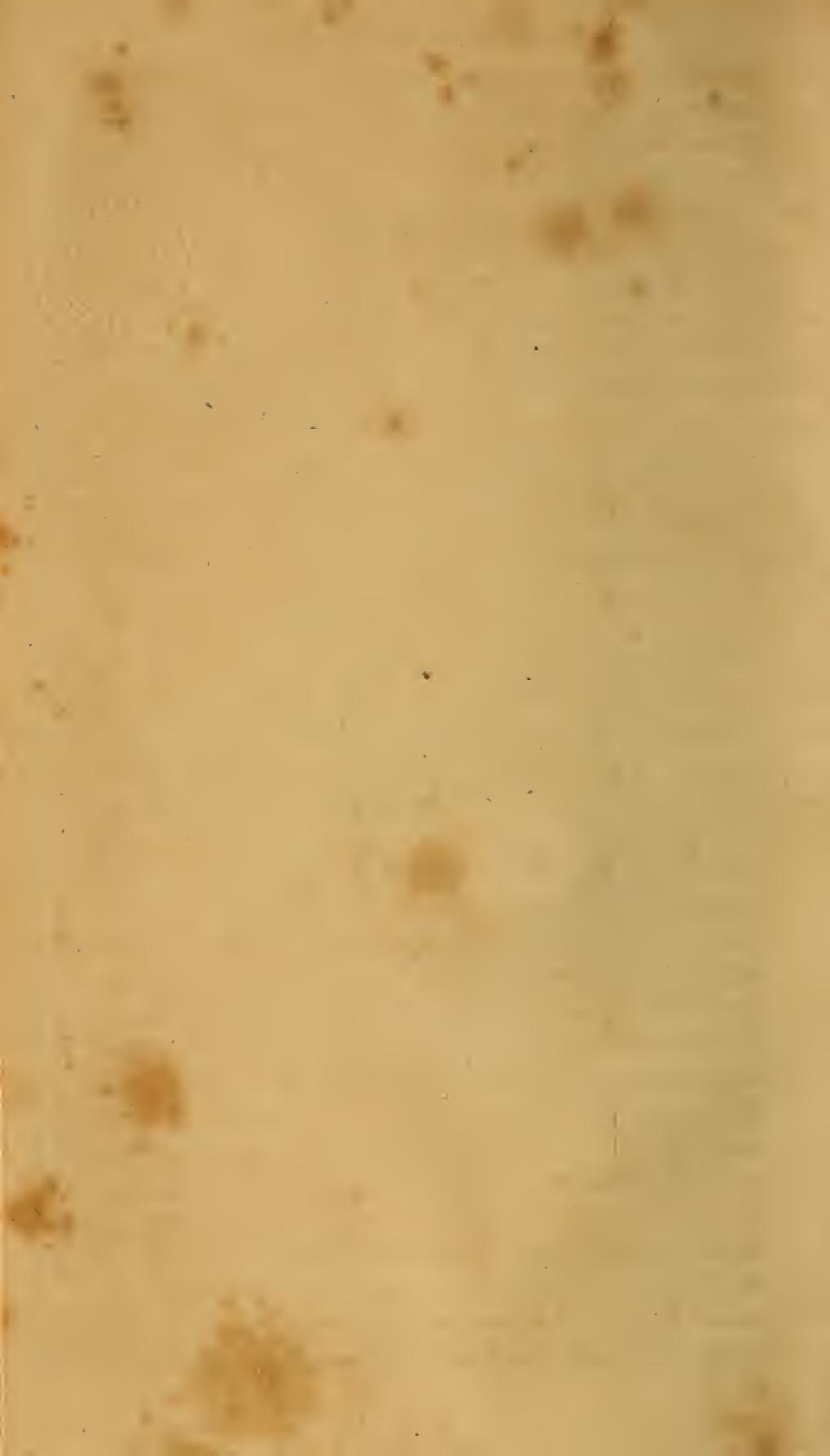
Such as we have said are the apparent magnitudes; but, because of the unequal refrangibility of light, all lucid points are dilated by the telescope, and in the focus of the object-glass possess a circular space whose breadth is about the 50th part of the aperture of the glass.

It is true, that towards the circumference the light is so rare as hardly to move the sense; but towards the middle, where it is of greater density, and is sensible enough, it makes a small lucid circle, whose breadth varies according to the splendor of the lucid point, but is generally about the 3d, or 4th, or 5th part of the breadth of the whole.

Let ABD represent the circle of the whole light; PQ the small circle of the denser and clearer light; C the centre of both; CA, CB, semi-diameters of the greater circle containing a right angle at C; ACBE the square comprehended under these semi-diameters; AB the diagonal of that square; EGH an hyperbola with the centre C and asymptotes CA, CB; PG a perpendicular erected from any point P of the line BC, and meeting the hyperbola in G, and the right lines AB, AE, in K and F: and the density of the light in any place P, will, by my computation, be as the line FG, and therefore at the centre infinite, but near the circumference very small. And the whole light within the small circle PQ is to the whole without as the area of the quadrilateral figure CAKP to the triangle PKB. And we are to understand the small circle PQ to be there terminated, where FG, the density of the light, begins to be less than what is required to move the sense.

Hence it was, that, at the distance of 191382 feet, a fire of 3 feet in diameter, through a telescope of 3 feet, appeared to Mr. *Picart* of 8" in breadth, when it should have appeared only of 3" 14"; and hence it is that the brighter fixed stars





appear through the telescope as of 5" or 6" in diameter, and that with a good full light; but with a fainter light they appear to run out to a greater breadth. Hence, likewise, it was that *Hevelius*, by diminishing the aperture of the telescope, did cut off a great part of the light towards the circumference, and brought the disk of the star to be more distinctly defined, which, though hereby diminished, did yet appear as of 5" or 6" in diameter. But Mr. *Huygens*, only by clouding the eye-glass with a little smoke, did so effectually extinguish this scattered light, that the fixed stars appeared as mere points, void of all sensible breadth. Hence also it was that Mr. *Huygens*, from the breadth of bodies interposed to intercept the whole light of the planets, reckoned their diameters greater than others have measured them by the micrometer; for the scattered light, which could not be seen before for the stronger light of the planet, when the planet is hid, appears every way farther spread. Lastly, from hence it is that the planets appear so small in the disk of the sun, being lessened by the dilated light. For to *Hevelius*, *Galletius*, and Dr. *Halley*, Mercury did not seem to exceed 12" or 15"; and Venus appeared to Mr. *Crabtrie* only 1' 3"; to *Horrox* but 1' 12"; though by the mensurations of *Hevelius* and *Hugenius* without the sun's disk, it ought to have been seen at least 1' 24". Thus the apparent diameter of the moon, which in 1684, a few days both before and after the sun's eclipse, was measured at the observatory of *Paris* 31' 30", in the eclipse itself did not seem to exceed 30' or 30' 05"; and therefore the diameters of the planets are to be diminished when without the sun, and to be augmented when within it, by some seconds. But the errors seem to be less than usual in the mensurations that are made by the micrometer. So from the diameter of the shadow, determined by the eclipses of the satellites, Mr. *Flamsted* found that the semi-diameter of Jupiter was to the greatest elongation of the outmost satellite as 1 to 24,903. Wherefore since that elongation is 8' 13", the diameter of Jupiter will be 39 $\frac{1}{2}$ "; and, rejecting the scattered light, the diameter found by the micrometer 40" or 41" will be reduced to 39 $\frac{1}{2}$ "; and the diameter of Saturn 21"

is to be diminished by the like correction, and to be reckoned 20", or something less. But (if I am not mistaken) the diameter of the sun, because of its stronger light, is to be diminished something more, and to be reckoned about 32', or 32' 6".

That bodies so different in magnitude should come so near to an analogy with their forces, is not without some mystery (p. 180).

It may be that the remoter planets, for want of heat, have not those metallic substances and ponderous minerals with which our earth abounds; and that the bodies of Venus and Mercury, as they are more exposed to the sun's heat, are also harder baked, and more compact.

For, from the experiment of the burning-glaſs, we see that the heat increases with the density of light; and this density increases in the reciprocal duplicate proportion of the distance from the sun; from whence the sun's heat in Mercury is proved to be sevenfold its heat in our summer seasons. But with this heat our water boils; and those heavy fluids, quicksilver and the spirit of vitriol, gently evaporate, as I have tried by the thermometer; and therefore there can be no fluids in Mercury but what are heavy, and able to bear a great heat, and from which substances of great density may be nourished.

And why not, if God has placed different bodies at different distances from the sun, so as the denser bodies always possess the nearer places, and each body enjoys a degree of heat suitable to its condition, and proper for its nourishment? From this consideration it will best appear that the weights of all the planets are one to another as their forces.

But I should be glad the diameters of the planets were more accurately measured; and that may be done, if a lamp, set at a great distance, is made to shine through a circular hole, and both the hole and the light of the lamp are so diminished that the spectrum may appear through the telescope just like the planet, and may be defined by the same measure: then the diameter of the hole will be to its distance from the objective glaſs as the true diameter of the planet to its distance from us. The light of the lamp may be diminished by the interposition either of pieces of cloth, or of smoked glaſs.

Of kin to the analogy we have been describing, there is another observed between the forces and the bodies attracted (p. 173, 174, 175). Since the action of the centripetal force upon the planets decreases in the duplicate proportion of the distance, and the periodic time increases in the sesquiplicate thereof, it is evident that the actions of the centripetal force, and therefore the periodic times, would be equal in equal planets at equal distances from the sun ; and in equal distances of unequal planets the total actions of the centripetal force would be as the bodies of the planets ; for if the actions were not proportional to the bodies to be moved, they could not equally retract these bodies from the tangents of their orbs in equal times : nor could the motions of the satellites of Jupiter be so regular, if it was not that the circum-solar force was equally exerted upon Jupiter and all its satellites in proportion of their several weights. And the same thing is to be said of Saturn in respect of its satellite, and of our earth in respect of the moon, as appears from cor. 2 and 3, prop. 65. And, therefore, at equal distances, the actions of the centripetal force are equal upon all the planets in proportion of their bodies, or of the quantities of matter in their several bodies ; and for the same reason must be the same upon all the particles of the same size of which the planet is composed ; for if the action was greater upon some sort of particles than upon others than in proportion to their quantity of matter, it would be also greater or less upon the whole planets not in proportion of the quantity only, but likewise of the sort of the matter more copiously found in one and more sparingly in another.

In such bodies as are found on our earth of very different sorts, I examined this analogy with great accuracy (p. 112, 113).

If the action of the circum-terrestrial force is proportional to the bodies to be moved, it will (by the second law of motion) move them with equal velocity in equal times, and will make all bodies let fall to descend through equal spaces in equal times, and all bodies hung by equal threads to vibrate in equal times. If the action of the force was greater, the times would be less; if that was less, these would be greater.

But it has been long ago observed by others, that (allowance being made for the small resistance of the air) all bodies descend through equal spaces in equal times; and, by the help of pendulums, that equality of times may be distinguished to great exactness.

I tried the thing in gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provided two equal wooden boxes. I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the centre of oscillation of the other. The boxes, hung by equal threads of 11 feet, made a couple of pendulums perfectly equal in weight and figure, and equally exposed to the resistance of the air: and, placing the one by the other, I observed them to play together forwards and backwards for a long while, with equal vibrations. And therefore (by cor. 1 and 6, prop. 24, book 2) the quantity of matter in the gold was to the quantity of matter in the wood as the action of the motive force upon all the gold to the action of the same upon all the wood; that is, as the weight of the one to the weight of the other.

And by these experiments, in bodies of the same weight, could have discovered a difference of matter less than the thousandth part of the whole.

Since the action of the centripetal force upon the bodies attracted is, at equal distances, proportional to the quantities of matter in those bodies, reason requires that it should be also proportional to the quantity of matter in the body attracting.

For all action is mutual, and (p. 14, 27, v. 1, by the third law of motion) makes the bodies mutually to approach one to the other, and therefore must be the same in both bodies. It is true that we may consider one body as attracting, another as attracted; but this distinction is more mathematical than natural. The attraction is really common of either to other, and therefore of the same kind in both.

And hence it is that the attractive force is found in both. The sun attracts Jupiter and the other planets; Jupiter attracts its satellites; and, for the same reason, the satellites act-

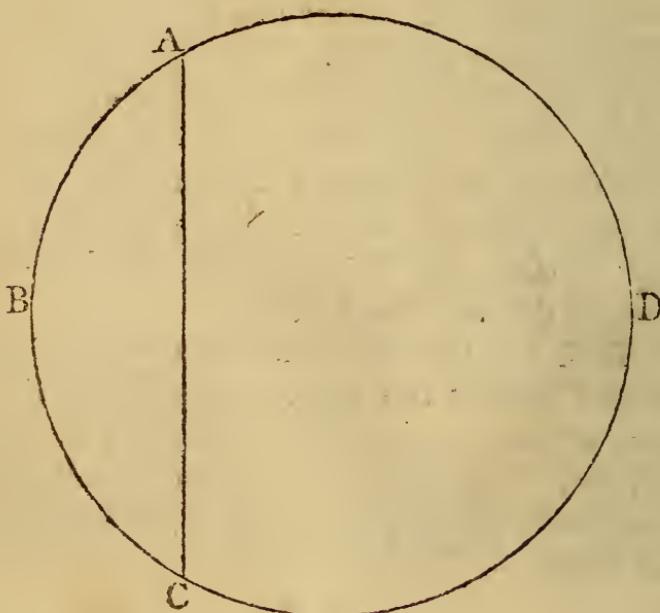
as well one upon another as upon Jupiter, and all the planets mutually one upon another.

And though the mutual actions of two planets may be distinguished and considered as two, by which each attracts the other, yet, as those actions are intermediate, they do not make two but one operation between two terms. Two bodies may be mutually attracted each to the other by the contraction of a cord interposed. There is a double cause of action, to wit, the disposition of both bodies, as well as a double action in so far as the action is considered as upon two bodies; but as betwixt two bodies it is but one single one. It is not one action by which the sun attracts Jupiter, and another by which Jupiter attracts the sun; but it is one action by which the sun and Jupiter mutually endeavour to approach each the other. By the action with which the sun attracts Jupiter, Jupiter and the sun endeavour to come nearer together (by the third law of motion); and by the action with which Jupiter attracts the sun, likewise Jupiter and the sun endeavour to come nearer together. But the sun is not attracted towards Jupiter by a twofold action, nor Jupiter by a twofold action towards the sun; but it is one single intermediate action, by which both approach nearer together.

Thus iron draws the load-stone (p. 27, vol. 1), as well as the load-stone draws the iron; for all iron in the neighbourhood of the load-stone draws other iron. But the action betwixt the load-stone and iron is single, and is considered as single by the philosophers. The action of iron upon the load-stone is, indeed, the action of the load-stone betwixt itself and the iron, by which both endeavour to come nearer together: and so it manifestly appears; for if you remove the load-stone, the whole force of the iron almost ceases.

In this sense it is that we are to conceive one single action to be exerted betwixt two planets, arising from the conspiring natures of both; and this action standing in the same relation to both, if it is proportional to the quantity of matter in the one, it will be also proportional to the quantity of matter in the other.

Perhaps it may be objected, that, according to this philosophy (p. 178), all bodies should mutually attract one another, contrary to the evidence of experiments in terrestrial bodies; but I answer, that the experiments in terrestrial bodies come to no account; for the attraction of homogeneous spheres near their surfaces are (by prop 72) as their diameters. Whence a sphere of one foot in diameter, and of a like nature to the earth, would attract a small body placed near its surface with a force 20000000 times less than the earth would do if placed near its surface; but so small a force could produce no sensible effect. If two such spheres were distant but by  $\frac{1}{4}$  of an inch, they would not, even in spaces void of resistance, come together by the force of their mutual attraction in less than a month's time; and less spheres will come together at a rate yet slower, viz. in the proportion of their diameters. Nay, whole mountains will not be sufficient to produce any sensible effect. A mountain of an hemispherical figure, three miles high, and six broad, will not, by its attraction, draw the pendulum two minutes out of the true perpendicular; and it is only in the great bodies of the planets that these forces are to be perceived, unless we may reason about smaller bodies in manner following.



Let ABCD (p. 27, vol. 1) represent the globe of the earth cut by any plane AC into two parts ACB, and ACD. The part ACB bearing upon the part ACD presses it with its whole weight; nor can the part ACD sustain this pressure, and continue unmoved, if it is not opposed by an equal contrary pressure. And therefore the parts equally press each other by their weights, that is, equally attract each other, according to the third law of motion; and, if separated and let go, would fall towards each other with velocities reciprocally as the bodies. All which we may try and see in the load-stone, whose attracted part does not propel the part attracting, but is only stopped and sustained thereby.

Suppose now that ACB represents some small body on the earth's surface; then, because the mutual attractions of this particle, and of the remaining part ACD of the earth towards each other, are equal, but the attraction of the particle towards the earth (or its weight) is as the matter of the particle (as we have proved by the experiment of the pendulums), the attraction of the earth towards the particle will likewise be as the matter of the particle; and therefore the attractive forces of all terrestrial bodies will be as their several quantities of matter.

The forces (p. 175), which are as the matter in terrestrial bodies of all forms, and therefore are not mutable with the forms, must be found in all sorts of bodies whatsoever, celestial as well as terrestrial, and be in all proportional to their quantities of matter, because among all there is no difference of substance, but of modes and forms only. But in the celestial bodies the same thing is likewise proved thus. We have shewn that the action of the circum-solar force upon all the planets (reduced to equal distances) is as the matter of the planets; that the action of the circum-jovial force upon the satellites of Jupiter observes the same law; and the same thing is to be said of the attraction of all the planets towards every planet: but thence it follows (by prop. 69) that their attractive forces are as their several quantities of matter.

As the parts of the earth mutually attract one another, so do those of all the planets. If Jupiter and its satellites were brought together, and formed into one globe, without doubt they would continue mutually to attract one another as before. And, on the other hand, if the body of Jupiter was broke into more globes, to be sure, these would no less attract one another than they do the satellites now. From these attractions it is that the bodies of the earth and all the planets effect a spherical figure, and their parts cohere, and are not dispersed through the æther. But we have before proved that these forces arise from the universal nature of matter (p. 177), and that, therefore, the force of any whole globe is made up of the several forces of all its parts. And from thence it follows (by cor. 3, prop. 74) that the force of every particle decreases in the duplicate proportion of the distance from that particle; and (by prop. 73 and 75) that the force of an entire globe, reckoning from the surface outwards, decreases in the duplicate, but, reckoning inwards, in the simple proportion of the distances from the centres, if the matter of the globe be uniform. And though the matter of the globe, reckoning from the centre towards the surface, is not uniform (p. 177, 178), yet the decrease in the duplicate proportion of the distance outwards would (by prop. 76) take place, provided that difformity is similar in places round about at equal distances from the centre. And two such globes will (by the same proposition) attract one the other with a force decreasing in the duplicate proportion of the distance between their centres.

Wherefore the absolute force of every globe is as the quantity of matter which the globe contains; but the motive force by which every globe is attracted towards another, and which, in terrestrial bodies, we commonly call their weight, is as the content under the quantities of matter in both globes applied to the square of the distance between their centres (by cor. 4, prop. 76), to which force the quantity of motion, by which each globe in a given time will be carried towards the other, is proportional. And the accelerative force, by which every globe according to its quantity of matter is attracted towards another, is as the quantity of matter in that other globe ap-

plied to the square of the distance between the centres of the two (by cor. 2, prop. 76) ; to which force, the velocity by which the attracted globe will, in a given time, be carried towards the other is proportional. And from these principles well understood, it will be now easy to determine the motions of the celestial bodies among themselves.

From comparing the forces of the planets one with another, we have above seen that the circum-solar does more than a thousand times exceed all the rest; but by the action of a force so great it is unavoidable but that all bodies within, nay, and far beyond, the bounds of the planetary system must descend directly to the sun, unless by other motions they are impelled towards other parts : nor is our earth to be excluded from the number of such bodies; for certainly the moon is a body of the same nature with the planets, and subject to the same attractions with the other planets, seeing it is by the circum-terrestrial force that it is retained in its orbit. But that the earth and moon are equally attracted towards the sun, we have above proved; we have likewise before proved that all bodies are subject to the said common laws of attraction. Nay, supposing any of those bodies to be deprived of its circular motion about the sun, by having its distance from the sun, we may find (by prop. 36) in what space of time it would in its descent arrive at the sun ; to wit, in half that periodic time in which the body might be revolved at one half of its former distance ; or in a space of time that is to the periodic time of the planet as 1 to  $4\sqrt{2}$ ; as that Venus in its descent would arrive at the sun in the space of 40 days, Jupiter in the space of two years and one month, and the earth and moon together in the space of 66 days and 19 hours. But, since no such thing happens, it must needs be, that those bodies are moved towards other parts (p. 3, vol. 1), nor is every motion sufficient for this purpose. To hinder such a descent, a due proportion of velocity is required. And hence depends the force of the argument drawn from the retardation of the motions of the planets. Unless the circum-solar force decreased in the duplicate ratio of their increasing slowness, the excess thereof would force

those bodies to descend to the sun; for instance, if the motion (*cæteris paribus*) was retarded by one half, the planet would be retained in its orb by one fourth of the former circum-solar force, and by the excess of the other three fourths would descend to the sun. And therefore the planets (Saturn, Jupiter, Mars, Venus, and Mercury) are not really retarded in their perigees, nor become really stationary, or regressive with slow motions. All these are but apparent, and the absolute motions, by which the planets continue to revolve in their orbits, are always direct, and nearly equable. But that such motions are performed about the sun, we have already proved; and therefore the sun, as the centre of the absolute motions, is quiescent. For we can by no means allow quiescence to the earth, lest the planets in their perigees should indeed be truly retarded, and become truly stationary and regressive, and so for want of motion should descend to the sun. But farther; since the planets (Venus, Mars, Jupiter, and the rest) by radii drawn to the sun describe regular orbits, and areas (as we have shewn) nearly and to sense proportional to the times, it follows (by prop. 3, and cor. 3, prop. 65) that the sun is moved with no notable force, unless perhaps with such as all the planets are equally moved with, according to their several quantities of matter, in parallel lines, and so the whole system is transferred in right lines. Reject that translation of the whole system, and the sun will be almost quiescent in the centre thereof. If the sun was revolved about the earth, and carried the other planets round about itself, the earth ought to attract the sun with a great force, but the circum-solar planets with no force producing any sensible effect, which is contrary to prop. 3, cor. 65. Add to this, that if hitherto the earth, because of the gravitation of its parts, has been placed by most authors in the lowermost region of the universe; now, for better reason, the sun possessed of a centripetal force exceeding our terrestrial gravitation a thousand times and more, ought to be depressed into the lowermost place, and to be held for the centre of the system. And thus the true disposition of the whole system will be more fully and more exactly understood.

Because the fixed stars are quiescent one in respect of another (p. 182, 183), we may consider the sun, earth, and planets, as one system of bodies carried hither and thither by various motions among themselves; and the common centre of gravity of all (by cor. 4 of the laws of motion) will either be quiescent, or move uniformly forward in a right line: in which case the whole system will likewise move uniformly forward in right lines. But this is an hypothesis hardly to be admitted; and, therefore, setting it aside, that common centre will be quiescent: and from it the sun is never far removed. The common centre of gravity of the sun and Jupiter falls on the surface of the sun; and though all the planets were placed towards the same parts from the sun with Jupiter the common centre of the sun and all of them would scarcely recede twice as far from the sun's centre; and, therefore, though the sun, according to the various situation of the planets, is variously agitated, and always wandering to and fro with a slow motion of libration, yet it never recedes one entire diameter of its own body from the quiescent centre of the whole system. But from the weights of the sun and planets above determined, and the situation of all among themselves, their common centre of gravity may be found; and, this being given, the sun's place to any supposed time may be obtained.

About the sun thus librated the other planets are revolved in elliptic orbits (p. 184), and, by radii drawn to the sun, describe areas nearly proportional to the times, as is explained in prop. 65. If the sun was quiescent, and the other planets did not act mutually one upon another, their orbits would be elliptic, and the areas exactly proportional to the times (by prop. 11, and cor. 1, prop. 13). But the actions of the planets among themselves, compared with the actions of the sun on the planets, are of no moment, and produce no sensible errors. And those errors are less in revolutions about the sun agitated in the manner but now described than if those revolutions were made about the sun quiescent (by prop. 66, and cor. prop. 68), especially if the focus of every orbit is placed in the common centre of gravity of

all the lower included planets; viz. the focus of the orbit of Mercury in the centre of the sun; the focus of the orbit of Venus in the common centre of gravity of Mercury and the sun; the focus of the orbit of the earth in the common centre of gravity of Venus, Mercury, and the sun; and so of the rest. And by this means the foci of the orbits of all the planets, except Saturn, will not be sensibly removed from the centre of the sun, nor will the focus of the orbit of Saturn recede sensibly from the common centre of gravity of Jupiter and the sun. And therefore astronomers are not far from the truth, when they reckon the sun's centre the common focus of all the planetary orbits. In Saturn itself the error thence arising does not exceed  $1' 45''$ . And if its orbit, by placing the focus thereof in the common centre of gravity of Jupiter and the sun, shall happen to agree better with the phænomena, from thence all that we have said will be farther confirmed.

If the sun was quiescent, and the planets did not act one on another, the aphelions and nodes of their orbits would likewise (by prop. 1, 11, and cor. prop. 13) be quiescent. And the longer axes of their elliptic orbits would (by prop. 15) be as the cubic roots of the squares of their periodic times: and therefore from the given periodic times would be also given. But those times are to be measured not from the equinoctial points, which are moveable, but from the first star of Aries. Put the semi-axis of the earth's orbit 100000, and the semi-axes of the orbits of Saturn, Jupiter, Mars, Venus, and Mercury, from their periodic times, will come out 953806, 520116, 152399, 72333, 38710 respectively. But from the sun's motion every semi-axis is increased (by prop. 60) by about one third of the distance of the sun's centre from the common centre of gravity of the sun and planet (p. 185, 186.) And from the actions of the exterior planets on the interior, the periodic times of the interior are something protracted, though scarcely by any sensible quantity; and their aphelions are transferred (by cor. 6 and 7, prop. 66) by very slow motions in *consequentia*. And on the like account the periodic times of all, especially of the exterior planets, will be pro-

longed by the actions of the comets, if any such there are, without the orb of Saturn, and the aphelions of all will be thereby carried forwards in *consequentia*. But from the progress of the aphelions the regress of the nodes follows (by cor. 11, 13, prop. 66). And if the plane of the ecliptic is quiescent, the regress of the nodes (by cor. 16, prop. 66) will be to the progress of the aphelion in every orbit as the regress of the nodes of the moon's orbit to the progress of its apogee nearly, that is, as about 10 to 21. But astronomical observations seem to confirm a very slow progress of the aphelions, and a regress of the nodes in respect of the fixed stars. And hence it is probable that there are comets in the regions beyond the planets, which, revolving in very eccentric orbits, quickly fly through their perihelion parts, and by an exceedingly slow motion in their aphelions spend almost their whole time in the regions beyond the planets; as we shall afterwards explain more at large.

The planets thus revolved about the sun (p. 197, 198, 199) may at the same time carry others revolving about themselves as satellites or moons, as appears by prop. 66. But from the action of the sun our moon must move with greater velocity, and, by a radius drawn to the earth, describe an area greater for the time; it must have its orbit less curve, and therefore approach nearer to the earth in the syzygies than in the quadratures, except in so far as the motion of eccentricity hinders those effects. For the eccentricity is greatest when the moon's apogee is in the syzygies, and least when the same is in the quadratures; and hence it is that the perigeon moon is swifter and nearer to us, but the apogee moon slower and farther from us, in the syzygies than in the quadratures. But farther; the apogee has a progressive and the nodes a regressive motion, both unequal. For the apogee is more swiftly progressive in its syzygies, more slowly regressive in its quadratures, and by the excess of its progress above its regress is yearly transferred in *consequentia*: but the nodes are quiescent in their syzygies, and most swiftly regressive in their quadratures. But farther, still, the greatest latitude of the moon is greater in its quadratures than in

its syzygies; and the mean motion swifter in the aphelion of the earth than in its perihelion. More inequalities in the moon's motion have not hitherto been taken notice of by astronomers: but all these follow from our principles in cor. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, prop. 66, and are known really to exist in the heavens. And this may be seen in that most ingenious, and, if I mistake not of all, the most accurate, hypothesis of Mr. *Horrox*, which Mr. *Flamsted* has fitted to the heavens; but the astronomical hypotheses are to be corrected in the motion of the nodes; for the nodes admit the greatest equation or prosthaphæresis in their octants, and this inequality is most conspicuous when the moon is in the nodes, and therefore also in the octants; and hence it was that *Tycho*, and others after him, referred this inequality to the octants of the moon, and made it menstrual; but the reasons by us adduced prove that it ought to be referred to the octants of the nodes, and to be made annual.

Beside those inequalities taken notice of by astronomers (p. 198, 235 to 237), there are yet some others, by which the moon's motions are so disturbed, that hitherto by no law could they be reduced to any certain regulation. For the velocities or horary motions of the apogee and nodes of the moon, and their equations, as well as the difference betwixt the greatest eccentricity in the syzygies and the least in the quadratures, and that inequality which we call the variation, in the progress of the year are augmented and diminished (by cor. 14, prop. 66) in the triplicate ratio of the sun's apparent diameter. Beside that, the variation is mutable nearly in the duplicate ratio of the time between the quadratures (by cor. 1 and 2, lem. 10, and cor. 16, prop. 66); and all those inequalities are something greater in that part of the orbit which respects the sun than in the opposite part, but by a difference that is scarcely or not at all perceptible.

By a computation (p. 209), which for brevity's sake I do not describe, I also find that the area which the moon by a radius drawn to the earth describes in the several equal moments of time is nearly as the sum of the number  $237\frac{3}{10}$ , and versed sine of the double distance of the moon from the

nearest quadrature in a circle whose radius is unity ; and therefore that the square of the moon's distance from the earth is as that sum divided by the horary motion of the moon. Thus it is when the variation in the octants is in its mean quantity ; but if the variation is greater or less, that versed sine must be augmented or diminished in the same ratio. Let astronomers try how exactly the distances thus found will agree with the moon's apparent diameters.

From the motions of our moon we may derive the motions of the moons or satellites of Jupiter and Saturn (p. 199) ; for the mean motion of the nodes of the outmost satellite of Jupiter is to the mean motion of the nodes of our moon in a proportion compounded of the duplicate proportion of the periodic time of the earth about the sun to the periodic time of Jupiter about the sun, and the simple proportion of the periodic time of the satellite about Jupiter to the periodic time of our moon about the earth (by cor. 16, prop. 66) : and therefore those nodes, in the space of a hundred years, are carried  $8^{\circ} 24'$  backwards, or in *antecedentia*. The mean motions of the nodes of the inner satellites are to the (mean) motion of (the nodes of) the outmost as their periodic times to the periodic time of this, by the same corollary, and are thence given. And the motion of the apsis of every satellite in *consequentia* is to the motion of its nodes in *antecedentia*, as the motion of the apogee of our moon to the motion of its nodes (by the same corollary), and is thence given. The greatest equations of the nodes and line of the apses of each satellite are to the greatest equations of the nodes and the line of the apses of the moon respectively as the motion of the nodes and line of the apses of the satellites in the time of one revolution of the first equations to the motion of the nodes and apogee of the moon in the time of one revolution of the last equations. The variation of a satellite seen from Jupiter is to the variation of our moon in the same proportion as the whole motions of their nodes respectively, during the times in which the satellite and our moon (after parting from) are revolved (again) to the sun, by the same corollary ; and therefore in the outmost satellite the variation does not exceed  $5'' 12''$ . From the small quantity

of those inequalities, and the slowness of the motions, it happens that the motions of the satellites are found to be so regular, that the more modern astronomers either deny all motion to the nodes, or affirm them to be very slowly regressive.

(P. 186). While the planets are thus revolved in orbits about remote centres, in the mean time they make their several rotations about their proper axes; the sun in 26 days; Jupiter in  $9^{\text{h}}. 56'$ ; Mars in  $24\frac{2}{3}^{\text{h}}.$ ; Venus in  $23^{\text{h}}.$ ; and that in planes not much inclined to the plane of the ecliptic, and according to the order of the signs, as astronomers determine from the spots or maculae that by turns present themselves to our sight in their bodies; and there is a like revolution of our earth performed in  $24^{\text{h}}.$ ; and those motions are neither accelerated nor retarded by the actions of the centripetal forces, as appears by cor. 22, prop. 66; and therefore of all others they are the most equitable and most fit for the mensuration of time; but those revolutions are to be reckoned equitable not from their return to the sun, but to some fixed star: for as the position of the planets to the sun is unequally varied, the revolutions of those planets from sun to sun are rendered unequable.

In like manner is the moon revolved about its axis by a motion most equitable in respect of the fixed stars, viz. in  $27^{\text{d}}. 7^{\text{h}}. 45'$ , that is, in the space of a sidereal month; so that this diurnal motion is equal to the mean motion of the moon in its orbit: upon which account, the same face of the moon always respects the centre about which this mean motion is performed, that is, the exterior focus of the moon's orbit nearly; and hence arises a deflection of the moon's face from the earth, sometimes towards the east, and other times towards the west, according to the position of the focus which it respects; and this deflection is equal to the equation of the moon's orbit, or to the difference betwixt its mean and true motions; and this is the moon's libration in longitude: but it is likewise affected with a libration in latitude arising from the inclination of the moon's axis to the plane of the orbit in which the moon is revolved about the earth; for that axis re-

tains the same position to the fixed stars nearly, and hence the poles present themselves to our view by turns, as we may understand from the example of the motion of the earth, whose poles, by reason of the inclination of its axis to the plane of the ecliptic, are by turns illuminated by the sun. To determine exactly the position of the moon's axis to the fixed stars, and the variation of this position, is a problem worthy of an astronomer.

By reason of the diurnal revolutions of the planets, the matter which they contain endeavours to recede from the axis of this motion; and hence the fluid parts rising higher towards the equator than about the poles (p. 187), would lay the solid parts about the equator under water, if those parts did not rise also (p. 187, 191, 192): upon which account the planets are something thicker about the equator than about the poles; and their equinoctial points (p. 197) thence become regressive; and their axes, by a motion of nutation, twice in every revolution, librate towards their ecliptics, and twice return again to their former inclination, as is explained in cor. 18, prop. 66; and hence it is that Jupiter, viewed through very long telescopes, does not appear altogether round (p. 191), but having its diameter that lies parallel to the ecliptic something longer than that which is drawn from north to south.

And from the diurnal motion and the attractions (p. 200, 204) of the sun and moon our sea ought twice to rise and twice to fall every day, as well lunar as solar (by cor. 19, 20, prop. 66), and the greatest height of the water to happen before the sixth hour of either day and after the twelfth hour preceding. By the slowness of the diurnal motion the flood is retracted to the twelfth hour; and by the force of the motion of reciprocation it is protracted and deferred till a time nearer to the sixth hour. But till that time is more certainly determined by the phænomena, choosing the middle between those extremes, why may we not conjecture the greatest height of the water to happen at the third hour? for thus the water will rise all that time in which the force of the lumi-  
naries to raise it is greater, and will fall all that time in which their force is less; viz. from the ninth to the third hour when

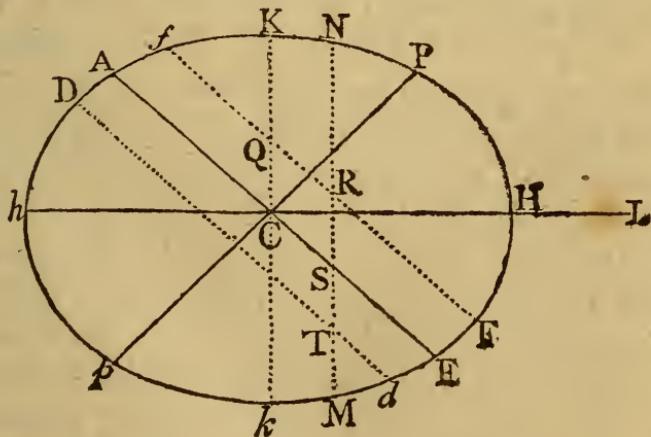
that force is greater, and from the third to the ninth when it is less. The hours I reckon from the appulse of each luminary to the meridian of the place, as well under as above the horizon ; and by the hours of the lunar day I understand the twenty-fourth parts of that time which the moon spends before it comes about again by its apparent diurnal motion to the meridian of the place which it left the day before.

But the two motions which the two luminaries raise will not appear distinguished, but will make a certain mixed motion. In the conjunction or opposition of the luminaries their forces will be conjoined, and bring on the greatest flood and ebb. In the quadratures the sun will raise the waters which the moon depresteth, and depress the waters which the moon raiseth ; and from the difference of their forces the smallest of all tides will follow. And because (as experience tells us) the force of the moon is greater than that of the sun, the greatest height of the water will happen about the third lunar hour. Out of the syzygies and quadratures the greatest tide which by the single force of the moon ought to fall out at the third lunar hour, and by the single force of the sun at the third solar hour, by the compounded forces of both must fall out in an intermediate time that approaches nearer to the third hour of the moon than to that of the sun ; and, therefore, while the moon is passing from the syzygies to the quadratures, during which time the third hour of the sun precedes the third of the moon, the greatest tide will precede the third lunar hour, and that by the greatest interval a little after the octants of the moon ; and by like intervals the greatest tide will follow the third lunar hour, while the moon is passing from the quadratures to the syzygies.

But the effects of the luminaries depend upon their distances from the earth; for when they are less distant their effects are greater, and when more distant their effects are less, and that in the triplicate proportion of their apparent diameters. Therefore it is that the sun in the winter time, being then in its perigee, has a greater effect, and makes the tides in the syzygies something greater, and those in the quadratures something less, *cæteris paribus*, than in the summer season ;

and every month the moon, while in the perigee, raiseth greater tides than at the distance of 15 days before or after, when it is in its apogee. Whence it comes to pass that two highest tides do not follow one the other in two immediately succeeding syzygies.

The effect of either luminary doth likewise depend upon its declination or distance from the equator; for if the luminary was placed at the pole, it would constantly attract all the parts of the waters, without any intension or remission of its action, and could cause no reciprocation of motion; and, therefore, as the luminaries decline from the equator towards either pole, they will by degrees lose their force, and on this account will excite lesser tides in the solstitial than in the equinoctial syzygies. But in the solstitial quadratures they will raise greater tides than in the quadratures about the equinoxes; because the effect of the moon, then situated in the equator, most exceeds the effect of the sun; therefore the greatest tides fall out in those syzygies, and the least in those quadratures, which happen about the time of both equinoxes; and the greatest tide in the syzygies is always succeeded by the least tide in the quadratures, as we find by experience. But because the sun is less distant from the earth in winter than in summer, it comes to pass that the greatest and least tides more frequently appear before than after the vernal equinox, and more frequently after than before the autumnal.



Moreover, the effects of the luminaries depend upon the latitudes of places. Let ApEP represent the earth on all sides covered with deep waters; C its centre; P, p, its poles; AE the equator; F any place without the equator; Ff the parallel of the place; Dd the correspondent parallel on the other side of the equator; L the place which the moon possessed three hours before; H the place of the earth directly under it; h the opposite place; K, k, the places at 90 degrees distance; CH, Ch, the greatest heights of the sea from the centre of the earth; and CK, Ck, the least heights: and if with the axes Hh, Kk, an ellipsis is described, and by the revolution of that ellipsis about its longer axis Hh a spheroid HPKhpk is formed, this spheroid will nearly represent the figure of the sea; and CF, Cf, CD, Cd, will represent the sea in the places F, f, D, d. But farther; if in the said revolution of the ellipsis any point N describes the circle NM, cutting the parallels Ff, Dd, in any places R, T, and the equator AE in S, CN will represent the height of the sea in all those places R, S, T, situated in this circle. Wherefore in the diurnal revolution of any place F the greatest flood will be in F, at the third hour after the appulse of the moon to the meridian above the horizon; and afterwards the greatest ebb in Q, at the third hour after the setting of the moon; and then the greatest flood in f, at the third hour after the appulse of the moon to the meridian under the horizon; and, lastly, the greatest ebb in Q, at the third hour after the rising of the moon; and the latter flood in f will be less than the preceding flood in F. For the whole sea is divided into two huge and hemispherical floods, one in the hemisphere KHkC on the north side, the other in the opposite hemisphere KHKc, which we may therefore call the northern and the southern floods: these floods, being always opposite the one to the other, come by turns to the meridians of all places after the interval of twelve lunar hours; and, seeing the northern countries partake more of the northern flood, and the southern countries more of the southern flood, thence arise tides alternately greater and less in all places without the equator in which the luminaries rise and set. But the greater tide will happen when the moon declines towards the

vertex of the place, about the third hour after the appulse of the moon to the meridian above the horizon ; and when the moon changes its declination, that which was the greater tide will be changed into a lesser; and the greatest difference of the floods will fall out about the times of the solstices, especially if the ascending node of the moon is about the first of Aries. So the morning tides in winter exceed those of the evening, and the evening tides exceed those of the morning in summer; at *Plymouth* by the height of one foot, but at *Bristol* by the height of 15 inches, according to the observations of *Colepress* and *Sturmy*.

But the motions which we have been describing suffer some alteration from that force of reciprocation which the waters [having once received] retain a little while by their *vis insita*; whence it comes to pass that the tides may continue for some time, though the actions of the luminaries should cease. This power of retaining the impressed motion lessens the difference of the alternate tides, and makes those tides which immediately succeed after the syzygies greater, and those which follow next after the quadratures less. And hence it is that the alternate tides at *Plymouth* and *Bristol* do not differ much more one from the other than by the height of a foot; or of 15 inches; and that the greatest tides of all at those ports are not the first but the third after the syzygies.

And, besides, all the motions are retarded in their passage through shallow channels, so that the greatest tides of all, in some straits and mouths of rivers, are the fourth, or even the fifth, after the syzygies.

It may also happen that the greatest tide may be the fourth or fifth after the syzygies, or fall out yet later, because the motions of the sea are retarded in passing through shallow places towards the shores; for so the tide arrives at the western coast of *Ireland* at the third lunar hour, and an hour or two after at the ports in the southern coast of the same island; as also at the islands *Cassiterides*, commonly *Sorlings*; then successively at *Falmouth*, *Plymouth*, *Portland*, the isle of *Wight*, *Winchester*, *Dover*, the mouth of the *Thames*, and *London Bridge*, spending twelve hours in this passage. But

farther ; the propagation of the tides may be obstructed even by the channels of the ocean itself, when they are not of depth enough ; for the flood happens at the third lunar hour in the *Canary* islands ; and at all those western coasts that lie towards the Atlantic ocean, as of *Ireland, France, Spain*, and all *Africa*, to the *Cape of Good Hope*, except in some shallow places, where it is impeded, and falls out later ; and in the straits of *Gibraltar*, where, by reason of a motion propagated from the mediterranean sea, it flows sooner. But, passing from those coasts over the breadth of the ocean to the coasts of *America*, the flood arrives first at the most eastern shores of *Braſil*, about the fourth or fifth lunar hour ; then at the mouth of the river of the *Amazons* at the sixth hour, but at the neighbouring islands at the fourth hour ; afterwards at the islands of *Bermudas* at the seventh hour, and at port *St. Augustin* in *Florida* at seven and a half. And therefore the tide is propagated through the ocean with a flower motion than it should be according to the course of the moon ; and this retardation is very necessary, that the sea at the same time may fall between *Braſil* and *New France*, and rise at the *Canary* islands, and on the coasts of *Europe* and *Africa*, and *vice versa* : for the sea cannot rise in one place but by falling in another. And it is probable that the *Pacific* sea is agitated by the same laws ; for in the coasts of *Chili* and *Peru* the highest flood is said to happen at the third lunar hour. But with what velocity it is thence propagated to the eastern coasts of *Japan*, the *Philippine* and other islands adjacent to *China*, I have not yet learned.

Farther ; it may happen (p. 204) that the tide may be propagated from the ocean through different channels towards the same port, and may pass quicker through some channels than through others, in which case the same tide, divided into two or more succeeding one another, may compound new motions of different kinds. Let us suppose one tide to be divided into two equal tides, the former whereof precedes the other by the space of six hours, and happens at the third or twenty-seventh hour from the appulse of the moon to the meridian of the port. If the moon at the time of this

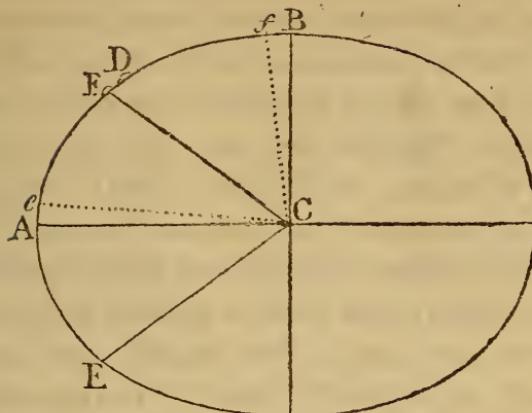
appulse to the meridian was in the equator, every six hours alternately there would arise equal floods, which, meeting with as many equal ebbs, would so balance one the other, that, for that day, the water would stagnate, and remain quiet. If the moon then declined from the equator, the tides in the ocean would be alternately greater and less, as was said ; and from hence two greater and two lesser tides would be alternately propagated towards that port. But the two greater floods would make the greatest height of the waters to fall out in the middle time betwixt both, and the greater and lesser floods would make the waters to rise to a mean height in the middle time between them ; and in the middle time between the two lesser floods the waters would rise to their least height. Thus in the space of twenty hours the waters would come, not twice, but once only to their greatest, and once only to their least height ; and their greatest height, if the moon declined towards the elevated pole, would happen at the sixth or thirtieth hour after the appulse of the moon to the meridian ; and when the moon changed its declination, this flood would be changed into an ebb.

Of all which we have an example in the port of *Batsham*, in the kingdom of *Tunquin*, in the latitude of  $20^{\circ} 50'$  north. In that port, on the day which follows after the passage of the moon over the equator, the waters stagnate ; when the moon declines to the north, they begin to flow and ebb, not twice, as in other ports, but once only every day ; and the flood happens at the setting, and the greatest ebb at the rising of the moon. This tide increaseth with the declination of the moon till the seventh or eighth day ; then for the seventh or eighth day following it decreaseth at the same rate as it had increased before, and ceaseth when the moon changeth its declination. After which the flood is immediately changed into an ebb ; and thenceforth the ebb happens at the setting and the flood at the rising of the moon, till the moon again changes its declination. There are two inlets from the ocean to this port ; one more direct and short between the island *Hainan* and the coast of *Quantung*, a province of *China* ; the other round about between the same island and the coast of

*Cochim*; and through the shorter passage the tide is sooner propagated to *Batsham*.

In the channels of rivers the influx and reflux depends upon the current of the rivers, which obstructs the ingress of the waters from the sea, and promotes their egress to the sea, making the ingress later and slower, and the egress sooner and faster; and hence it is that the reflux is of longer duration than the influx, especially far up the rivers, where the force of the sea is less. So *Sturmy* tells us, that in the river *Avon*, three miles below *Bristol*, the water flows only five hours, but ebbs seven; and without doubt the difference is yet greater above *Bristol*, as at *Careham* or the *Bath*. This difference does likewise depend upon the quantity of the flux and reflux; for the more vehement motion of the sea near the syzygies of the luminaries more easily overcoming the resistance of the rivers, will make the ingress of the water to happen sooner and to continue longer, and will therefore diminish this difference. But while the moon is approaching to the syzygies, the rivers will be more plentifully filled, their currents being obstructed by the greatness of the tides, and therefore will something more retard the reflux of the sea a little after than a little before the syzygies. Upon which account the lowest tides of all will not happen in the syzygies, but prevent them a little; and I observed above that the tides before the syzygies were also retarded by the force of the sun; and from both causes conjoined the retardation of the tides will be both greater and sooner before the syzygies. All which I find to be so, by the tide-tables which *Flamsteed* has composed from a great many observations.

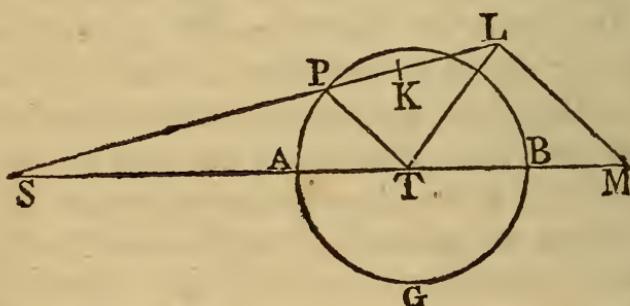
By the laws we have been describing, the times of the tides are governed; but the greatness of the tides depends upon the greatness of the seas. Let C represent the centre of the earth, EADB the oval figure of the seas, CA the longer semi-axis of this oval, CB the shorter insisting at right angles upon the former, D the middle point between A and B, and ECF or eCf the angle at the centre of the earth, subtended by the breadth of the sea that terminates in the shores E, F, or e, f. Now, supposing that the point A is in the mid-



dle between the points E, F, and the point D in the middle between the points e, f, if the difference of the heights CA, CB, represent the quantity of the tide in a very deep sea surrounding the whole earth, the excess of the height CA above the height CE or CF will represent the quantity of the tide in the middle of the sea EF, terminated by the shores E, F; and the excess of the height Ce above the height Cf will nearly represent the quantity of the tide on the shores f of the same sea. Whence it appears that the tides are far less in the middle of the sea than at the shores; and that the tides at the shores are nearly as EF (p. 243, 244), the breadth of the sea not exceeding a quadrantial arc. And hence it is that near the equator, where the sea between *Africa* and *America* is narrow, the tides are far less than towards either side in the temperate zones, where the seas are extended wider; or on almost all the shores of the *Pacific* sea, as well towards *America* as towards *China*, and within as well as without the tropics; and that in islands in the middle of the sea they scarcely rise higher than two or three feet, but on the shores of great continents are three or four times greater, and above, especially if the motions propagated from the ocean are by degrees contracted into a narrow space, and the water, to fill and empty the bays alternately, is forced to flow and ebb, with great violence through shallow places; as *Plymouth* and *Chepstow Bridge* in *England*, at the mount of *St. Michael* and town of *Avranches* in *Normandy*, and at *Cambaia* and *Pegu* in the *East Indies*. In which places, the

sea, hurried in and out with great violence, sometimes lays the shores under water, sometimes leaves them dry, for many miles. Nor is the force of the influx and efflux to be broke till it has raised or depressed the water to forty or fifty feet and more. Thus also long and shallow straits that open to the sea with mouths wider and deeper than the rest of their channel (such as those about *Britain*, and the *Magellanic Straits* at the eastern entry) will have a greater flood and ebb, or will more intend and remit their course, and therefore will rise higher and be depressed lower. On the coast of *South America* it is said that the *Pacific* sea in its reflux sometimes retreats two miles, and gets out of sight of those that stand on shore. Whence in these places the floods will be also higher; but in deeper waters the velocity of influx and efflux is always less, and therefore the ascent and descent is so too. Nor in such places is the ocean known to ascend to more than six, eight, or ten feet. The quantity of the ascent I compute in the following manner.

Let *S* represent the sun, *T* the earth (p. 205, 206), *P* the moon, *PAGB* the moon's orbit. In *SP* take *SK* equal to *ST*

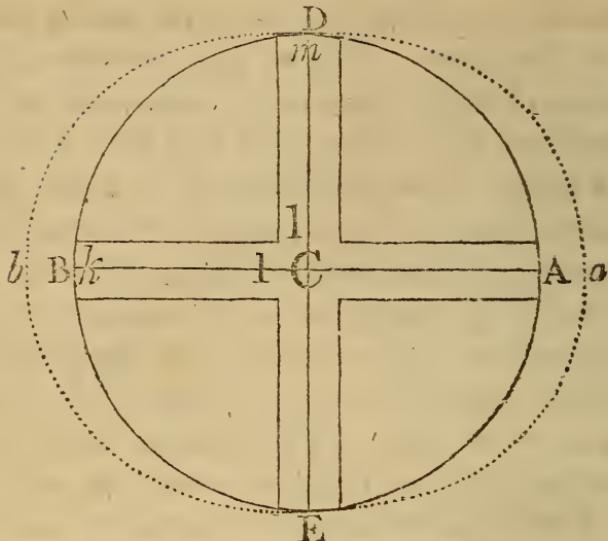


and *SL* to *SK* in the duplicate ratio of *SK* to *SP*. Parallel to *PT* draw *LM*; and, supposing the mean quantity of the circum-solar force directed towards the earth to be represented by the distance *ST* or *SK*, *SL* will represent the quantity thereof directed towards the moon. But that force is compounded of the parts *SM*, *LM*; of which the force *LM*, and that part of *SM* which is represented by *TM*, do disturb the motion of the moon (as appears from prop. 66, and its corollaries). In so far as the earth and moon are revolved about

their common centre of gravity, the earth will be liable to the action of the like forces. But we may refer the sums as well of the forces as of the motions to the moon, and represent the sums of the forces by the lines TM and ML, which are proportional to them. The force LM, in its mean quantity, is to the force by which the moon may be revolved in an orbit, about the earth quiescent, at the distance PT in the duplicate ratio of the moon's periodic time about the earth to the earth's periodic time about the sun (by cor. 17, prop. 66); that is, in the duplicate ratio of  $27^{\text{d}}. 7^{\text{h}}. 43'$  to  $365^{\text{d}}. 6^{\text{h}}. 9'$ , or as 1000 to 178725, or 1 to  $178\frac{2}{45}$ . The force by which the moon may be revolved in its orb about the earth in rest, at the distance PT of  $60\frac{1}{2}$  semi-diameters of the earth, is to the force by which it may revolve in the same time at the distance of 60 semi-diameters as  $60\frac{1}{2}$  to 60; and this force is to the force of gravity with us as 1 to  $60 \times 60$  nearly; and therefore the mean force ML is to the force of gravity at the surface of the earth as  $1 \times 60\frac{1}{2}$  to  $60 \times 60 \times 178\frac{2}{45}$ , or 1 to 638092,6. Whence the force TM will be also given from the proportion of the lines TM, ML. And these are the forces of the sun, by which the moon's motions are disturbed.

If from the moon's orbit (p. 240) we descend to the earth's surface, those forces will be diminished in the ratio of the distances  $60\frac{1}{2}$  and 1; and therefore the force LM will then become 38604600 times less than the force of gravity. But this force acting equally every where upon the earth, will scarcely effect any change on the motion of the sea, and therefore may be neglected in the explication of that motion. The other force TM, in places where the sun is vertical, or in their nadir, is triple the quantity of the force ML, and therefore but 12868200 times less than the force of gravity.

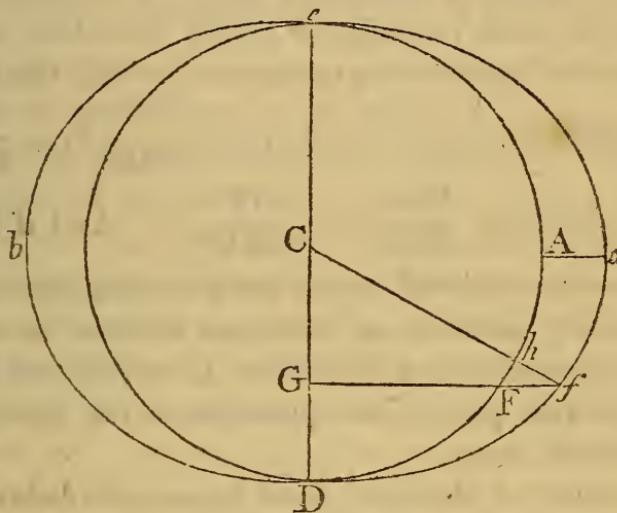
Suppose now ADBE to represent the spherical surface of the earth, aDbE the surface of the water overspreading it, C the centre of both, A the place to which the sun is vertical, B the place opposite; D, E, places at 90 degrees distance from the former; ACEmlk a right angled cylindric canal passing through the earth's centre. The force TM in any place is as the distance of the place from the plane DE, on which



a line from A to C insists at right angles, and therefore in the part of the canal which is represented by EC<sub>m</sub> is of no quantity, but in the other part ACl<sub>k</sub> is as the gravity at the several heights; for in descending towards the centre of the earth, gravity is (by prop. 73) every where as the height; and therefore the force TM drawing the water upwards will diminish its gravity in the leg ACl<sub>k</sub> of the canal in a given ratio: upon which account the water will ascend in this leg, till its defect of gravity is supplied by its greater height; nor will it rest in an equilibrium till its total gravity becomes equal to the total gravity in EC<sub>m</sub>, the other leg of the canal. Because the gravity of every particle is as its distance from the earth's centre, the weight of the whole water in either leg will increase in the duplicate ratio of the height; and therefore the height of the water in the leg ACl<sub>k</sub> will be to the height thereof in the leg Cl<sub>m</sub>E in the subduplicate ratio of the number 12868201 to 12868200, or in the ratio of the number 25623053 to the number 25623052, and the height of the water in the leg EC<sub>m</sub> to the difference of the heights, as 25623052 to 1. But the height in the leg EC<sub>m</sub> is of 19615800 Paris feet, as has been lately found by the mensuration of the *French*; and, therefore, by the preceding analogy, the difference of the heights comes out 9 $\frac{1}{5}$  inches of the Paris foot; and the sun's force will make the height of the sea at A to exceed the height of the same at E by 9

inches. And though the water of the canal ACEmlk be supposed to be frozen into a hard and solid consistence, yet the heights thereof at A and E, and all other intermediate places, would still remain the same.

Let Aa (in the following figure) represent that excess of height of .9 inches at A, and hf the excess of height at any other place h; and upon DC let fall the perpendicular fG, meeting the globe of the earth in F; and because the distance of the sun is so great that all the right lines drawn thereto may be considered as parallel, the force TM in any place f will be to the same force in the place A as the fine FG to the radius AC. And, therefore, since those forces tend to the sun in the direction of parallel lines, they will generate the



parallel heights Ff, Aa, in the same ratio; and therefore the figure of the water Dfaeb will be a spheroid made by the revolution of an ellipsis about its longer axis ab. And the perpendicular height fh will be to the oblique height Ff as fG to fC, or as FG to AC: and therefore the height fh is to the height Aa in the duplicate ratio of FG to AC, that is, in the ratio of the versed sine of double the angle DCf to double the radius, and is thence given. And hence to the several moments of the apparent revolution of the sun about the earth we may infer the proportion of the ascent and descent of the waters at any given place under the equator, as well as of the diminution of that ascent and descent, whether arising

from the latitude of places or from the sun's declination; viz. that on account of the latitude of places, the ascent and descent of the sea is in all places diminished in the duplicate ratio of the co-sines of latitude; and on account of the sun's declination, the ascent and descent under the equator is diminished in the duplicate ratio of the co-sine of declination. And in places without the equator the half sum of the morning and evening ascents (that is, the mean ascent) is diminished nearly in the same ratio.

Let S and L respectively represent the forces of the sun and moon placed in the equator, and at their mean distances from the earth; R the radius; T and V the versed sines of double the complements of the sun and moon's declinations to any given time; D and E the mean apparent diameters of the sun and moon: and, supposing F and G to be their apparent diameters to that given time, their forces to raise the tides un-

der the equator will be, in the syzygies,  $\frac{VG^3}{2RE^3} L + \frac{TF^3}{2RD^3} S$ ;

in the quadratures,  $\frac{VG^3}{2RE^3} L - \frac{TF^3}{2RD^3} S$ . And if the same ratio is likewise observed under the parallels, from observations accurately made in our northern climates we may determine the proportion of the forces L and S; and then by means of this rule predict the quantities of the tides to every syzygy and quadrature.

At the mouth of the river *Avon*, three miles below *Bristol* (p. 241, 242, 243, 244), in spring and autumn, the whole ascent of the water in the conjunction or opposition of the luminaries (by the observation of *Sturmy*) is about 45 feet, but in the quadratures only 25. Because the apparent diameters of the luminaries are not here determined, let us assume them in their mean quantities, as well as the moon's declination in the equinoctial quadratures in its mean quantity, that is,  $23\frac{1}{2}^\circ$ ; and the versed sine of double its complement will be 1682, supposing the radius to be 1000. But the declinations of the sun in the equinoxes and of the moon in the syzygies are of no quantity, and the versed sines of double the complements are each 2000. Whence those forces become L + S

in the fzygyes, and  $\frac{1682}{2000} L - S$  in the quadratures, respectively proportional to the heights of the tides of 45 and 25 feet, or of 9 and 5 paces. And, therefore, multiplying the extremes and the means, we have  $5L + 5S = \frac{15138}{2000} L - 9S$ , or  $L = \frac{28000}{5138} S = 5\frac{5}{11} S$ .

But farther; I remember to have been told, that in summer the ascent of the sea in the fzygyes is to the ascent thereof in the quadratures as about 5 to 4. In the solstices themselves it is probable that the proportion may be something less, as about 6 to 5; whence it would follow that  $L$  is  $=$

$5\frac{1}{3} S$  [for then the proportion is  $\frac{1682}{2000} L + \frac{1682}{2000} S : L = \frac{1682}{2000} S :: 6 : 5$ ]. Till we can more certainly determine the proportion from observation, let us assume  $L = 5\frac{1}{3} S$ ; and since the heights of the tides are as the forces which excite them, and the force of the sun is able to raise the tides to the height of nine inches, the moon's force will be sufficient to raise the same to the height of four feet. And if we allow that this height may be doubled, or perhaps tripled, by that force of reciprocation which we observe in the motion of the waters, and by which their motion once begun is kept up for some time, there will be force enough to generate all that quantity of tides which we really find in the ocean.

Thus we have seen that these forces are sufficient to move the sea. But, so far as I can observe, they will not be able to produce any other effect sensible on our earth; for since the weight of one grain in 4000 is not sensible in the nicest balance; and the sun's force to move the tides is 12868200 less than the force of gravity; and the sum of the forces of both moon and sun, exceeding the sun's force only in the ratio of  $6\frac{1}{3}$  to 1, is still 2032890 times less than the force of gravity; it is evident that both forces together are 500 times less than what is required sensibly to increase or diminish the weight of any body in a balance. And, therefore, they will not sensibly move any suspended body; nor will they pro-

duce any sensible effect on pendulums, barometers, bodies swimming in stagnant water, or in the like statical experiments. In the atmosphere, indeed, they will excite such a flux and reflux as they do in the sea, but with so small a motion that no sensible wind will be thence produced.

If the effects of both moon and sun in raising the tides (p. 245), as well as their apparent diameters, were equal among themselves, their absolute forces would (by cor. 14, prop. 66) be as their magnitudes. But the effect of the moon is to the effect of the sun as about  $5\frac{1}{3}$  to 1; and the moon's diameter less than the sun's in the ratio of  $31\frac{1}{2}$  to  $32\frac{1}{5}$ , or of 45 to 46. Now the force of the moon is to be increased in the ratio of the effect directly, and in the triplicate ratio of the diameter inversely. Whence the force of the moon compared with its magnitude will be to the force of the sun compared with its magnitude in the ratio compounded of  $5\frac{1}{3}$  to 1, and the triplicate of 45 to 46 inversely, that is, in the ratio of about  $5\frac{7}{10}$  to 1. And therefore the moon, in respect of the magnitude of its body, has an absolute centripetal force greater than the sun in respect of the magnitude of its body in the ratio of  $5\frac{7}{10}$  to 1, and is therefore more dense in the same ratio.

In the time of  $27^d. 7^h. 43'$ , in which the moon makes its revolution about the earth, a planet may be revolved about the sun at the distance of 18,954 diameters of the sun from the sun's centre, supposing the mean apparent diameter of the sun to be  $32\frac{1}{5}'$ ; and in the same time the moon may be revolved about the earth at rest, at the distance of 30 of the earth's diameters. If in both cases the number of diameters was the same, the absolute circum-terrestrial force would (by cor. 2, prop. 72) be to the absolute circum-solar force as the magnitude of the earth to the magnitude of the sun. Because the number of the earth's diameters is greater in the ratio of 30 to 18,954, the body of the earth will be less in the triplicate of that ratio, that is, in the ratio of  $3\frac{2}{9}\frac{8}{9}$  to 1. Wherefore the earth's force, for the magnitude of its body, is to the sun's force, for the magnitude of its body, as  $3\frac{2}{9}\frac{8}{9}$  to 1; and consequently the earth's density to the sun's will be in the same

ratio. Since, then, the moon's density is to the sun's density as  $5\frac{7}{10}$  to 1, the moon's density will be to the earth's density as  $5\frac{7}{10}$  to  $3\frac{2}{9}$ , or as 23 to 16. Wherefore since the moon's magnitude is to the earth's magnitude as about 1 to  $41\frac{1}{2}$ , the moon's absolute centripetal force will be to the earth's absolute centripetal force as about 1 to 29, and the quantity of matter in the moon to the quantity of matter in the earth in the same ratio. And hence the common centre of gravity of the earth and moon is more exactly determined than hitherto has been done; from the knowledge of which we may now infer the moon's distance from the earth with greater accuracy. But I would rather wait till the proportion of the bodies of the moon and earth one to the other is more exactly defined from the phænomena of the tides, hoping that in the mean time the circumference of the earth may be measured from more distant stations than any body has yet employed for this purpose.

Thus I have given an account of the system of the planets. As to the fixed stars, the smallness of their annual parallax proves them to be removed to immense distances from the system of the planets: that this parallax is less than one minute is most certain; and from thence it follows that the distance of the fixed stars is above 360 times greater than the distance of Saturn from the sun. Such as reckon the earth one of the planets, and the sun one of the fixed stars, may remove the fixed stars to yet greater distances by the following arguments: from the annual motion of the earth there would happen an apparent transposition of the fixed stars, one in respect of another, almost equal to their double parallax; but the greater and nearer stars, in respect of the more remote, which are only seen by the telescope, have not hitherto been observed to have the least motion. If we should suppose that motion to be but less than 20", the distance of the nearer fixed stars would exceed the mean distance of Saturn by above 2000 times. Again; the disk of Saturn, which is only 17" or

18" in diameter, receives but about  $\frac{1}{2100000000}$  of the sun's light; for so much less is that disk than the whole spherical

surface of the orb of Saturn. Now if we suppose Saturn to reflect about  $\frac{1}{4}$  of this light, the whole light reflected from its illuminated hemisphere will be about  $\frac{1}{4200000000}$  of the whole light emitted from the sun's hemisphere; and, therefore, since light is rarefied in the duplicate ratio of the distance from the luminous body, if the sun was  $10000\sqrt{42}$  times more distant than Saturn, it would yet appear as lucid as Saturn now does without its ring, that is, something more lucid than a fixed star of the first magnitude. Let us, therefore, suppose that the distance from which the sun would shine as a fixed star exceeds that of Saturn by about 100,000 times, and its apparent diameter will be  $7^{\circ} 16' 16''$ . and its parallax arising from the annual motion of the earth  $13'''$ : and so great will be the distance, the apparent diameter, and the parallax of the fixed stars of the first magnitude, in bulk and light equal to our sun. Some may, perhaps, imagine that a great part of the light of the fixed stars is intercepted and lost in its passage through so vast spaces, and upon that account pretend to place the fixed stars at nearer distances; but at this rate the remoter stars could be scarcely seen. Suppose, for example, that  $\frac{3}{4}$  of the light perish in its passage from the nearest fixed stars to us; then  $\frac{1}{4}$  will twice perish in its passage through a double space, thrice through a triple, and so forth. And, therefore, the fixed stars that are at a double distance will be 16 times more obscure, viz. 4 times more obscure on account of the diminished apparent diameter; and, again, 4 times more on account of the lost light. And, by the same argument, the fixed stars at a triple distance will be  $9 \times 4 \times 4$ , or 144 times more obscure; and those at a quadruple distance will be  $16 \times 4 \times 4 \times 4$ , or 1024 times more obscure; but so great a diminution of light is no ways consistent with the phænomena and with that hypothesis which places the fixed stars at different distances.

The fixed stars being, therefore, at such vast distances from one another (p. 254, 255), can neither attract each other sensibly, nor be attracted by our sun. But the comets must unavoidably be acted on by the circum-solar force; for as the

comets were placed by astronomers above the moon, because they were found to have no diurnal parallax, so their annual parallax is a convincing proof of their descending into the regions of the planets. For all the comets which move in a direct course, according to the order of the signs, about the end of their appearance become more than ordinarily slow, or retrograde, if the earth is between them and the sun; and more than ordinarily swift if the earth is approaching to a heliocentric opposition with them. Whereas, on the other hand, those which move against the order of the signs, towards the end of their appearance, appear swifter than they ought to be if the earth is between them and the sun; and slower, and perhaps retrograde, if the earth is in the other side of its orbit. This is occasioned by the motion of the earth in different situations. If the earth go the same way with the comet, with a swifter motion, the comet becomes retrograde; if with a slower motion, the comet becomes slower however; and if the earth move the contrary way, it becomes swifter; and by collecting the differences between the slower and swifter motions, and the sums of the more swift and retrograde motions, and comparing them with the situation and motion of the earth from whence they arise, I found, by means of this parallax, that the distances of the comets at the time they cease to be visible to the naked eye are always less than the distance of Saturn, and generally even less than the distance of Jupiter.

The same thing may be collected from the curvature of the way of the comets (p. 256). These bodies go on nearly in great circles while their motion continues swift; but about the end of their course, when that part of their apparent motion which arises from the parallax bears a greater proportion to their whole apparent motion, they commonly deviate from those circles; and when the earth goes to one side, they deviate to the other; and this deflection, because of its corresponding with the motion of the earth, must arise chiefly from the parallax; and the quantity thereof is so considerable, as, by my computation, to place the disappearing comets a good deal lower than Jupiter. Whence it follows, that, when

they approach nearer to us in their perigees and perihelions, they often descend below the orbits of Mars and the inferior planets.

Moreover, this nearness of the comets is confirmed by the annual parallax of the orbit, in so far as the same is pretty nearly collected by the supposition that the comets move uniformly in right lines. The method of collecting the distance of a comet according to this hypothesis from four observations (first attempted by *Kepler*, and perfected by Dr. *Wallis* and Sir *Christopher Wren*) is well known; and the comets reduced to this regularity generally pass through the middle of the planetary region. So the comets of the years 1607 and 1618, as their motions are defined by *Kepler*, passed between the sun and the earth; that of the year 1664 below the orbit of Mars; and that in 1680 below the orbit of Mercury, as its motion was defined by Sir *Christopher Wren* and others. By a like rectilinear hypothesis, *Hevelius* placed all the comets about which we have any observations below the orbit of Jupiter. It is a false notion, therefore, and contrary to astronomical calculation, which some have entertained, who, from the regular motion of the comets, either remove them into the regions of the fixed stars, or deny the motion of the earth; whereas their motions cannot be reduced to perfect regularity, unless we suppose them to pass through the regions near the earth in motion; and these are the arguments drawn from the parallax, so far as it can be determined without an exact knowledge of the orbits and motions of the comets.

The near approach of the comets is farther confirmed from the light of their heads (p. 257 to 260); for the light of a celestial body, illuminated by the sun, and receding to remote parts, is diminished in the quaduplicate proportion of the distance; to wit, in one duplicate proportion on account of the increase of the distance from the sun; and in another duplicate proportion on account of the decrease of the apparent diameter. Hence it may be inferred, that Saturn being at a double distance, and having its apparent diameter nearly half of that of Jupiter, must appear about 16 times more obscure; and that, if its distance were 4 times greater, its light would

be 256 times less; and therefore would be hardly perceptible to the naked eye. But now the comets often equal Saturn's light, without exceeding him in their apparent diameters. So the comet of the year 1668, according to Dr. *Hooke*'s observations, equalled in brightness the light of a fixed star of the first magnitude; and its head, or the star in the middle of the coma, appeared, through a telescope of 15 feet, as lucid as Saturn near the horizon; but the diameter of the head was only 25"; that is, almost the same with the diameter of a circle equal to Saturn and his ring. The coma or hair surrounding the head was about ten times as broad; namely,  $4\frac{1}{6}$  min. Again; the least diameter of the hair of the comet of the year 1682, observed by Mr. *Flamsted* with a tube of 16 feet, and measured with the micrometer, was 2' 0"; but the nucleus, or star in the middle, scarcely possessed the tenth part of this breadth, and was therefore only 11" or 12" broad; but the light and clearness of its head exceeded that of the year 1680, and was equal to that of the stars of the first or second magnitude. Moreover, the comet of the year 1665, in *April*, as *Hevelius* informs us, exceeded almost all the fixed stars in splendor, and even Saturn itself, as being of a much more vivid colour; for this comet was more lucid than that which appeared at the end of the foregoing year, and was compared to the stars of the first magnitude. The diameter of the coma was about 6'; but the nucleus, compared with the planets by means of a telescope, was plainly less than Jupiter, and was sometimes thought less, sometimes equal to the body of Saturn within the ring. To this breadth add that of the ring, and the whole face of Saturn will be twice as great as that of the comet, with a light not at all more intense; and therefore the comet was nearer to the sun than Saturn. From the proportion of the nucleus to the whole head found by these observations, and from its breadth, which seldom exceeds 8' or 12', it appears that the stars of the comets are most commonly of the same apparent magnitude as the planets; but that their light may be compared oftentimes with that of Saturn, and sometimes exceeds it. And hence it is certain that in their perihelia their distances can scarcely be greater than

that of Saturn. At twice that distance, the light would be four times less, which besides by its dim paleness would be as much inferior to the light of Saturn as the light of Saturn is to the splendor of Jupiter: but this difference would be easily observed. At a distance ten times greater, their bodies must be greater than that of the sun; but their light would be 100 times fainter than that of Saturn. And at distances still greater, their bodies would far exceed the sun; but, being in such dark regions, they must be no longer visible. So impossible is it to place the comets in the middle regions between the sun and fixed stars, accounting the sun as one of the fixed stars; for certainly they would receive no more light there from the sun than we do from the greatest of the fixed stars.

So far we have gone without considering that obscuration which comets suffer from that plenty of thick smoke which compasseth their heads, and through which the heads always shew dull as through a cloud; for by how much the more a body is obscured by this smoke, by so much the more near it must be allowed to come to the sun, that it may vie with the planets in the quantity of light which it reflects; whence it is probable that the comets descend far below the orbit of Saturn, as we proved before from their parallax. But, above all, the thing is evinced from their tails, which must be owing either to the sun's light reflected from a smoke arising from them, and dispersing itself through the æther, or to the light of their own heads.

In the former case we must shorten the distance of the comets, lest we be obliged to allow that the smoke arising from their heads is propagated through such a vast extent of space, and with such a velocity of expansion, as will seem altogether incredible; in the latter case the whole light of both head and tail must be ascribed to the central nucleus. But, then, if we suppose all this light to be united and condensed within the disk of the nucleus, certainly the nucleus will by far exceed Jupiter itself in splendor, especially when it emits a very large and lucid tail. If, therefore, under a less apparent diameter, it reflects more light, it must be much more illuminated

by the sun, and therefore much nearer to it. So the comet that appeared Dec. 12 and 15, O.S. *Anno* 1679, at the time it emitted a very shining tail, whose splendor was equal to that of many stars like Jupiter, if their light were dilated and spread through so great a space, was, as to the magnitude of its nucleus, less than Jupiter (as Mr. *Flamsted* observed), and therefore was much nearer to the sun: nay, it was even less than Mercury. For on the 17th of that month, when it was nearer to the earth, it appeared to *Cassini* through a telescope of 35 feet a little less than the globe of Saturn. On the 8th of this month, in the morning, Dr. *Halley* saw the tail, appearing broad and very short, and as if it rose from the body of the sun itself, at that time very near its rising. Its form was like that of an extraordinary bright cloud; nor did it disappear till the sun itself began to be seen above the horizon. Its splendor, therefore, exceeded the light of the clouds till the sun rose, and far surpassed that of all the stars together, as yielding only to the immediate brightness of the sun itself. Neither Mercury, nor Venus, nor the moon itself, are seen so near the rising sun. Imagine all this dilated light collected together, and to be crowded into the orbit of the comet's nucleus which was less than Mercury; by its splendor, thus increased, becoming so much more conspicuous, it will vastly exceed Mercury, and therefore must be nearer to the sun. On the 12th and 15th of the same month, this tail, extending itself over a much greater space, appeared more rare; but its light was still so vigorous as to become visible when the fixed stars were hardly to be seen, and soon after to appear like a fiery beam shining in a wonderful manner. From its length, which was 40 or 50 degrees, and its breadth of 2 degrees, we may compute what the light of the whole must be.

This near approach of the comets to the sun is confirmed from the situation they are seen in when their tails appear most resplendent; for when the head passes by the sun, and lies hid under the solar rays, very bright and shining tails, like fiery beams, are said to issue from the horizon; but afterwards, when the head begins to appear, and is got far-

ther from the sun, that splendor always decreases, and turns by degrees into a paleness like to that of the milky way, but much more sensible at first; after that vanishing gradually. Such was that most resplendent comet described by *Aristotle*, lib. 1. Meteor. 6. “The head thereof could not be seen, because it set before the sun, or at least was hid under the sun’s rays; but the next day it was seen as well as might be; for, having left the sun but a very little way, it set immediately after it; and the scattered light of the head obscured by the too great splendour (of the tail) did not yet appear. But afterwards (says *Aristotle*), when the splendour of the tail was now diminished, (the head of) the comet recovered its native brightness. And the splendour of its tail reached now to a third part of the heavens (that is, to  $60^{\circ}$ ). It appeared in the winter season, and, rising to *Orion*’s girdle, there vanished away.” Two comets of the same kind are described by *Justin*, lib. 37, which, according to his account, “ shined so bright, that the whole heaven seemed to be on fire; and by their greatness filled up a fourth part of the heavens, and by their splendour exceeded that of the sun.” By which last words a near position of these bright comets and the rising or setting sun is intimated (p. 296, 297). We may add to these the comet of the year 1101 or 1106, “ the star of which was small and obscure (like that of 1680); but the splendour arising from it extremely bright, reaching like a fiery beam to the east and north,” as *Hevelius* has it from *Simeon*, the monk of *Durham*. It appeared at the beginning of *February* about the evening in the south-west. From this and from the situation of the tail we may infer that the head was near the sun. *Matthew Paris* says, “ it was about one cubit from the sun; from the third [or rather the sixth] to the ninth hour sending out a long stream of light.” The comet of 1264, in *July*, or about the solstice, preceded the rising sun, sending out its beams with a great light towards the west as far as the middle of the heavens; and at the beginning it ascended a little above the horizon: but as the sun went forwards it retired every day farther from the horizon, till it passed by the very middle of the heavens. It is said to have been at the beginning large

and bright, having a large coma, which decayed from day to day. It is described in *Append. Matth. Paris. Hist. Ang.* after this manner : “ *An. Christi* 1265, there appeared a comet so wonderful, that none then living had ever seen the like ; for, rising from the east with a great brightness, it extended itself with a great light as far as the middle of the hemisphere towards the west.” The Latin original being somewhat barbarous and obscure, it is here subjoined. *Ab oriente enim cum magno fulgore surgens, usque ad medium hemisphœrii versus occidentem, omnia perlucide pertrahebat.*

“ In the year 1401 or 1402, the sun being got below the horizon, there appeared in the west a bright and shining comet, sending out a tail upwards, in splendor like a flame of fire, and in form like a spear, darting its rays from west to east. When the sun was sunk below the horizon, by the lustre of its own rays it enlightened all the borders of the earth, not permitting the other stars to shew their light, or the shades of night to darken the air, because its light exceeded that of the others, and extended itself to the upper part of the heavens, flaming,” &c. *Hist. Byzant. Duc. Mich. Nepot.* From the situation of the tail of this comet, and the time of its first appearance, we may infer that the head was then near the sun, and went farther from him every day ; for that comet continued three months. In the year 1527, *Aug. 11*, about four in the morning, there was seen almost throughout *Europe* a terrible comet in *Leo*, which continued flaming an hour and a quarter every day. It rose from the east, and ascended to the south and west to a prodigious length. It was most conspicuous to the north, and its cloud (that is, its tail) was very terrible ; having, according to the fancies of the vulgar, the form of an arm a little bent holding a sword of a vast magnitude. In the year 1618, in the end of *November*, there began a rumour, that there appeared about sun-rising a bright beam, which was the tail of a comet whose head was yet concealed within the brightness of the solar rays. On *Nov. 24*, and from that time, the comet itself appeared with a bright light, its head and tail being extremely resplendent. The length of the tail, which was at first 20 or 30 deg.

increased till *December* 9, when it arose to 75 deg. but with a light much more faint and dilute than at the beginning. In the year 1668, *March* 5, N. S. about seven in the evening, *P. Valent. Eftancius*, being in *Brasil*, saw a comet near the horizon in the south-west. Its head was small, and scarcely discernible, but its tail extremely bright and resplendent, so that the reflection of it from the sea was easily seen by those who stood upon the shore. This great splendor lasted but three days, decreasing very remarkably from that time. The tail at the beginning extended itself from west to south, and in a situation almost parallel to the horizon, appearing like a shining beam 23 deg. in length. Afterwards, the light decreasing, its magnitude increased till the comet ceased to be visible; so that *Cassini*, at *Bologna*, saw it (*Mar. 10, 11, 12*) rising from the horizon 32 deg. in length. In *Portugal* it is said to have taken up a fourth part of the heavens (that is, 45 deg.), extending itself from west to east with a notable brightness; though the whole of it was not seen, because the head in this part of the world always lay hid below the horizon. From the increase of the tail it is plain that the head receded from the sun, and was nearest to it at the beginning, when the tail appeared brightest.

To all these we may add the comet of 1680, whose wonderful splendor at the conjunction of the head with the sun was above described. But so great a splendor argues the comets of this kind to have really passed near the fountain of light, especially since the tails never shine so much in their opposition to the sun; nor do we read that fiery beams have ever appeared there.

Lastly, the same thing is inferred (p. 259 to 261) from the light of the heads increasing in the recesses of the comets from the earth towards the sun, and decreasing in their return from the sun towards the earth; for so the last comet of the year 1665 (by the observation of *Hevelius*), from the time that it was first seen, was always losing of its apparent motion, and therefore had already passed its perigee; yet the splendor of its head was daily increasing, till, being hid by the sun's rays, the comet ceased to appear. The comet of the year 1683 (by

the observation of the same *Hevelius*), about the end of *July*, when it first appeared, moved at a very slow rate, advancing only about 40 or 45 minutes in its orbit in a day's time. But from that time its diurnal motion was continually upon the increase till *September 4*, when it arose to about 5 degrees; and therefore in all this interval of time the comet was approaching to the earth. Which is likewise proved from the diameter of its head measured with a micrometer; for, *August* the 6th, *Hevelius* found it only 6' 5", including the coma; which, *September 2*, he observed 9' 7". And therefore its head appeared far less about the beginning than towards the end of its motion, though about the beginning, because nearer to the sun, it appeared far more lucid than towards the end, as the same *Hevelius* declares. Wherefore in all this interval of time, on account of its recess from the sun, it decreased in splendor, notwithstanding its access towards the earth. The comet of the year 1618, about the middle of *December*, and that of the year 1680, about the end of the same month, did both move with their greatest velocity, and were therefore then in their perigees; but the greatest splendor of their heads was seen two weeks before, when they had just got clear of the sun's rays; and the greatest splendor of their tails a little more early, when yet nearer to the sun. The head of the former comet, according to the observations of *Cysatus*, *Dec. 1*, appeared greater than the stars of the first magnitude; and, *Dec. 16* (being then in its perigee), of a small magnitude, and the splendor or clearness was much diminished. *Jan. 7*, *Kepler*, being uncertain about the head, left off observing. *Dec. 12*, the head of the last comet was seen and observed by *Flamsted* at the distance of 9 degrees from the sun, which a star of the third magnitude could hardly have been. *December 15* and *17*, the same appeared like a star of the third magnitude, its splendor being diminished by the bright clouds near the setting sun. *Dec. 26*, when it moved with the greatest swiftness, and was almost in its perigee, it was inferior to *Os Pegasi*, a star of the third magnitude. *Jan. 3*, it appeared like a star of the fourth; *Jan. 9*, like a star of the fifth. *Jan. 13*, it disappear-

ed, by reason of the brightness of the moon, which was then in its increase. *Jan. 25*, it was scarcely equal to the stars of the seventh magnitude. If we take equal times on each hand of the perigee, the heads placed at remote distances would have shined equally before and after, because of their equal distances from the earth. That in one case they shined very bright, and in the other vanished, is to be ascribed to the nearness of the sun in the first case, and his distance in the other; and from the great difference of the light in these two cases we infer its great nearness in the first of them; for the light of the comets uses to be regular, and to appear greatest when their heads move the swiftest, and are therefore in their perigees; excepting in so far as it is increased by their nearness to the sun.

From these things I at last discovered why the comets frequent so much the region of the sun. If they were to be seen in the regions a great way beyond Saturn, they must appear oftener in those parts of the heavens that are opposite to the sun; for those which are in that situation would be nearer to the earth, and the interposition of the sun would obscure the others: but, looking over the history of comets, I find that four or five times more have been seen in the hemisphere toward the sun than in the opposite hemisphere; besides, without doubt, not a few which have been hid by the light of the sun; for comets descending into our parts neither emit tails, nor are so well illuminated by the sun, as to discover themselves to our naked eyes, till they are come nearer to us than Jupiter. But the far greater part of that spherical space, which is described about the sun with so small an interval, lies on that side of the earth which regards the sun, and the comets in that greater part are more strongly illuminated, as being for the most part nearer to the sun: besides, from the remarkable eccentricity of their orbits, it comes to pass that their lower apsides are much nearer to the sun than if their revolutions were performed in circles concentric to the sun.

Hence also we understand why the tails of the comets, while their heads are descending towards the sun, always appear

short and rare, and are seldom said to have exceeded 15 or 20 deg. in length ; but in the recesses of the heads from the sun often shine like fiery beams, and soon after reach to 40, 50, 60, 70 deg. in length, or more. This great splendor and length of the tails arises from the heat which the sun communicates to the comet as it passes near it. And thence, I think, it may be concluded, that all the comets that have had such tails have passed very near the sun.

Hence also we may collect that the tails arise from the atmospheres of the heads (p. 287 to 288) : but we have had three several opinions about the tails of comets ; for some will have it that they are nothing else but the beams of the sun's light transmitted through the comets' heads, which they suppose to be transparent ; others, that they proceed from the refraction which light suffers in passing from the comet's head to the earth ; and, lastly, others, that they are a sort of clouds or vapour constantly rising from the comets' heads, and tending towards the parts opposite to the sun. The first is the opinion of such as are yet unacquainted with optics ; for the beams of the sun are not seen in a darkened room, but in consequence of the light that is reflected from them by the little particles of dust and smoke which are always flying about in the air ; and hence it is that in air impregnated with thick smoke they appear with greater brightness, and are more faintly and more difficultly seen in a finer air ; but in the heavens, where there is no matter to reflect the light, they are not to be seen at all. Light is not seen as it is in the beams, but as it is thence reflected to our eyes ; for vision is not made but by rays falling upon the eyes, and therefore there must be some reflecting matter in those parts where the tails of comets are seen ; and so the argument turns upon the third opinion ; for that reflecting matter can be no where found but in the place of the tail, because otherwise, since all the celestial spaces are equally illuminated by the sun's light, no part of the heavens could appear with more splendor than another. The second opinion is liable to many difficulties. The tails of comets are never seen variegated with those colours which ever use to be inseparable from refraction ; and the

distinct transmission of the light of the fixed stars and planets to us is a demonstration that the æther or celestial medium is not endowed with any refractive power. For as to what is alledged that the fixed stars have been sometimes seen by the *Egyptians* environed with a coma or capillitium, because that has but rarely happened, it is rather to be ascribed to a casual refraction of clouds, as well as the radiation and scintillation of the fixed stars to the refractions both of the eyes and air; for upon applying a telescope to the eye, those radiations and scintillations immediately disappear. By the tremulous agitation of the air and ascending vapours, it happens that the rays of light are alternately turned aside from the narrow space of the pupil of the eye; but no such thing can have place in the much wider aperture of the object-glass of a telescope; and hence it is that a scintillation is occasioned in the former case which ceases in the latter; and this cessation in the latter case is a demonstration of the regular transmission of light through the heavens without any sensible refraction. But, to obviate an objection that may be made from the appearing of no tail in such comets as shine but with a faint light, as if the secondary rays were then too weak to affect the eyes, and for this reason it is that the tails of the fixed stars do not appear, we are to consider that by the means of telescopes the light of the fixed stars may be augmented above an hundred fold, and yet no tails are seen; that the light of the planets is yet more copious without any tail, but that comets are seen sometimes with huge tails, when the light of their heads is but faint and dull; for so it happened in the comet of the year 1680, when in the month of December it was scarcely equal in light to the stars of the second magnitude, and yet emitted a notable tail, extending to the length of  $40^{\circ}$ ,  $50^{\circ}$ ,  $60^{\circ}$ , or  $70^{\circ}$ , and upwards; and afterwards, on the 27th and 28th of January, the head appeared but as a star of the seventh magnitude; but the tail (as was said above), with a light that was sensible enough, though faint, was stretched out to 6 or 7 degrees in length, and with a languishing light that was more difficultly seen, even to  $12^{\circ}$  and upwards. But on the 9th and 10th of February, when

to the naked eye the head appeared no more, I saw through a telescope the tail of  $2^{\circ}$  in length. But farther; if the tail was owing to the refraction of the celestial matter, and did deviate from the opposition of the sun, according as the figure of the heavens requires, that deviation, in the same places of the heavens, should be always directed towards the same parts: but the comet of the year 1680, December 28<sup>d</sup>. 8 $\frac{1}{2}$ <sup>h</sup>. P. M. at *London*, was seen in *Pisces*  $8^{\circ} 41'$ , with latitude north  $28^{\circ} 6'$ , while the sun was in *Capricorn*  $18^{\circ} 26'$ . And the comet of the year 1577, December 29, was in *Pisces*  $8^{\circ} 41'$ , with latitude north  $28^{\circ} 40'$ ; and the sun, as before, in about *Capricorn*  $18^{\circ} 26'$ . In both cases the situation of the earth was the same, and the comet appeared in the same place of the heavens; yet in the former case the tail of the comet (as well by my observations as by the observations of others) deviated from the opposition of the sun towards the north by an angle of  $4\frac{1}{2}$  degrees, whereas in the latter there was (according to the observation of *Tycho*) a deviation of  $21$  degrees towards the south. The refraction, therefore, of the heavens being thus disproved, it remains that the phænomena of the tails of comets must be derived from some reflecting matter. That vapours sufficient to fill such immense spaces may arise from the comets' atmospheres, may be easily understood by what follows.

It is well known that the air near the surface of our earth possesses a space about 1200 times greater than water of the same weight; and therefore a cylindric column of air 1200 feet high is of equal weight with a cylinder of water of the same breadth, and but one foot high. But a cylinder of air reaching to the top of the atmosphere is of equal weight with a cylinder of water about 33 feet high; and therefore if from the whole cylinder of air the lower part of 1200 feet high is taken away, the remaining upper part will be of equal weight with a cylinder of water 32 feet high. Wherefore at the height of 1200 feet, or two furlongs, the weight of the incumbent air is less, and consequently the rarity of the compressed air greater, than near the surface of the earth in the ratio of 33 to 32. And, having this ratio, we may compute

the rarity of the air in all places whatsoever (by the help of cor. prop. 2, book 2), supposing the expansion thereof to be reciprocally proportional to its compression ; and this proportion has been proved by the experiments of *Hooke* and others. The result of the computation I have set down in the following table, in the first column of which you have the height of the air in miles, whereof 4000 make a semi-diameter of the earth ; in the second the compression of the air, or the incumbent weight ; in the third its rarity or expansion, supposing gravity to decrease in the duplicate ratio of the distances from the earth's centre. And the *Latin* numeral characters are here used for certain numbers of ciphers, as 0, xvii 1224 for 0,0000000000000001224, and 26956 xv for 26956000000000000000.

AIR's

Height.	Compression.	Expansion.
033		1
5	17,8515	1,8486
10	9 6717	3,4151
20	2,852	11,571
40	0,2525	136,83
400	0,xvii 1224	26956 xv
4000	0,cv 4465	73907 cii
40000	0,cxcii 1628	20263 clxxxix
400000	0,ccx 7895	41798 ccvii
4000000	0,cxxii 9878	33414 ccix
Infinite.	0,ccxii 6041	54622 ccix

But from this table it appears that the air, in proceeding upwards, is rarefied in such manner, that a sphere of that air which is nearest to the earth, of but one inch in diameter, if dilated with that rarefaction which it would have at the height of one semi-diameter of the earth, would fill all the planetary regions as far as the sphere of Saturn, and a great way beyond ; and at the height of ten semi-diameters of the earth would fill up more space than is contained in the whole heavens on this side the fixed stars, according to the preceding computation of their distance. And though, by reason of the far greater thickness of the atmospheres of comets, and the great quantity of the circum-solar centripetal force, it may

happen that the air in the celestial spaces, and in the tails of comets, is not so vastly rarefied, yet from this computation it is plain that a very small quantity of air and vapour is abundantly sufficient to produce all the appearances of the tails of comets; for that they are indeed of a very notable rarity appears from the shining of the stars through them. The atmosphere of the earth, illuminated by the sun's light, though but of a few miles in thickness, obscures and extinguishes the light not only of all the stars, but even of the moon itself; whereas the smallest stars are seen to shine through the immense thickness of the tails of comets, likewise illuminated by the sun, without the least diminution of their splendor.

*Kepler* ascribes the ascent of the tails of comets to the atmospheres of their heads, and their direction towards the parts opposite to the sun to the action of the rays of light carrying along with them the matter of the comets' tails; and without any great incongruity we may suppose that, in so free spaces, so fine a matter as that of the æther may yield to the action of the rays of the sun's light, though those rays are not able sensibly to move the gross substances in our parts, which are clogged with so palpable a resistance. Another author thinks that there may be a sort of particles of matter endowed with a principle of levity as well as others are with a power of gravity; that the matter of the tails of comets may be of the former sort, and that its ascent from the sun may be owing to its levity; but, considering the gravity of terrestrial bodies is as the matter of the bodies, and therefore can be neither more nor less in the same quantity of matter, I am inclined to believe that this ascent may rather proceed from the rarefaction of the matter of the comets' tails. The ascent of smoke in a chimney is owing to the impulse of the air with which it is entangled. The air rarefied by heat ascends, because its specific gravity is diminished, and in its ascent carries along with it the smoke with which it is engaged. And why may not the tail of a comet rise from the sun after the same manner? for the sun's rays do not act any way upon the mediums which they pervade but by reflection and refraction; and those reflecting particles heated

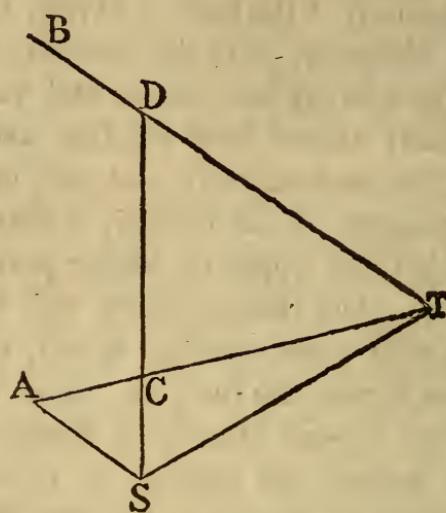
by this action, heat the matter of the æther which is involved with them. That matter is rarefied by the heat which it acquires, and because by this rarefaction the specific gravity, with which it tended towards the sun before, is diminished, it will ascend therefrom like a stream, and carry along with it the reflecting particles of which the tail of the comet is composed ; the impulse of the sun's light, as we have said, promoting the ascent.

But that the tails of comets do arise from their heads (p. 288), and tend towards the parts opposite to the sun, is farther confirmed from the laws which the tails observe ; for, lying in the planes of the comets' orbits which pass through the sun, they constantly deviate from the opposition of the sun towards the parts which the comets' heads in their progress along those orbits have left ; and to a spectator placed in those planes they appear in the parts directly opposite to the sun ; but as the spectator recedes from those planes, their deviation begins to appear, and daily becomes greater. And the deviation, *cæteris paribus*, appears less when the tail is more oblique to the orbit of the comet, as well as when the head of the comet approaches nearer to the sun ; especially if the angle of deviation is estimated near the head of the comet. Farther ; the tails which have no deviation appear straight, but the tails which deviate are likewise bended into a certain curvature ; and this curvature is greater when the deviation is greater, and is more sensible when the tail, *cæteris paribus*, is longer ; for in the shorter tails the curvature is hardly to be perceived. And the angle of deviation is less near the comet's head, but greater towards the other end of the tail, and that because the lower side of the tail regards the parts from which the deviation is made, and which lie in a right line drawn out infinitely from the sun through the comet's head. And the tails that are longer and broader, and shine with a stronger light, appear more resplendent and more exactly defined on the convex than on the concave side. Upon which accounts it is plain that the phænomena of the tails of comets depend upon the motions of their heads, and by no means upon the places of the heavens in which their

heads are seen; and that, therefore, the tails of the comets do not proceed from the refraction of the heavens, but from their own heads, which furnish the matter that forms the tail; for as in our air the smoke of a heated body ascends either perpendicularly if the body is at rest, or obliquely if the body is moved obliquely, so in the heavens, where all the bodies gravitate towards the sun, smoke and vapour must (as we have already said) ascend from the sun, and either rise perpendicularly if the smoking body is at rest, or obliquely if the body, in the progress of its motion, is always leaving those places from which the upper or higher parts of the vapours had risen before. And that obliquity will be less where the vapour ascends with more velocity, to wit, near the smoking body, when that is near the sun; for there the force of the sun by which the vapour ascends is stronger. But because the obliquity is varied, the column of vapour will be incurvated; and because the vapour in the preceding side is something more recent, that is, has ascended something more lately from the body, it will therefore be something more dense on that side, and must on that account reflect more light, as well as be better defined; the vapour on the other side languishing by degrees, and vanishing out of sight.

But it is none of our present business to explain the causes of the appearances of nature. Let those things which we have last said be true or false, we have at least made out, in the preceding discourse, that the rays of light are directly propagated from the tails of comets in right lines through the heavens, in which those tails appear to the spectators wherever placed; and consequently the tails must ascend from the heads of the comets towards the parts opposite to the sun. And from this principle we may determine anew the limits of their distances in manner following. Let S represent the sun, T the earth, STA the elongation of a comet from the sun, and ATB the apparent length of its tail; and because the light is propagated from the extremity of the tail in the direction of the right line TB, that extremity must lie somewhere in the line TB. Suppose it in D, and join DS cutting TA in C. Then, because the tail is always stretched out towards the parts

nearly opposite to the sun, and therefore the sun, the head of the comet, and the extremity of the tail, lie in a right line, the comet's head will be found in C. Parallel to TB draw SA,



meeting the line TA in A, and the comet's head C must necessarily be found between T and A, because the extremity of the tail lies somewhere in the infinite line TB; and all the lines SD which can possibly be drawn from the point S to the line TB must cut the line TA somewhere between T and A. Wherefore the distance of the comet from the earth cannot exceed the interval TA, nor its distance from the sun the interval SA beyond, or ST on this side the sun. For instance; the elongation of the comet of 1680 from the sun, Dec. 12, was  $9^\circ$ , and the length of its tail  $35^\circ$  at least. If, therefore, a triangle TSA is made, whose angle T is equal to the elongation  $9^\circ$ , and angle A equal to ATB, or to the length of the tail, viz.  $35^\circ$ , then SA will be to ST, that is, the limit of the greatest possible distance of the comet from the sun to the semi-diameter of the *orbis magnus*, as the fine of the angle T to the fine of the angle A, that is, as about 3 to 11. And therefore the comet at that time was less distant from the sun than by  $\frac{3}{11}$  of the earth's distance from the sun, and consequently either was within the orb of Mercury, or between that orb and the earth. Again, Dec. 21, the elongation of the comet from the sun was  $32\frac{2}{5}^\circ$ , and the length of its tail  $70^\circ$ .

Wherefore as the sine of  $32\frac{2}{5}^{\circ}$  to the sine of  $70^{\circ}$ , that is, as 4 to 7, so was the limit of the comet's distance from the sun to the distance of the earth from the sun, and consequently the comet had not then got without the orb of Venus. *Dec. 28*, the elongation of the comet from the sun was  $55^{\circ}$ , and the length of its tail  $56^{\circ}$ ; and therefore the limit of the comet's distance from the sun was not yet equal to the distance of the earth from the same, and consequently the comet had not then got without the earth's orbit. But from its parallax we find that its egress from the orbit happened about *Jan. 5*, as well as that it had descended far within the orbit of Mercury. Let us suppose it to have been in its perihelion *Dec. the 8th*, when it was in conjunction with the sun; and it will follow that in the journey from its perihelion to its exit out of the earth's orbit it had spent 28 days; and consequently that in the 26 or 27 days following, in which it ceased to be farther seen by the naked eye, it had scarcely doubled its distance from the sun; and by limiting the distances of other comets by the like arguments, we come at last to this conclusion,---that all comets, during the time in which they are visible by us, are within the compass of a spherical space described about the sun as a centre, with a radius double, or at most triple, of the distance of the earth from the sun.

And hence it follows that the comets, during the whole time of their appearance unto us, being within the sphere of activity of the circum-solar force, and therefore agitated by the impulse of that force, will (by cor. 1, prop. 13, for the same reason as the planets) be made to move in conic sections that have one focus in the centre of the sun, and, by radii drawn to the sun, to describe areas proportional to the times; for that force is propagated to an immense distance, and will govern the motions of bodies far beyond the orbit of Saturn.

There are three hypotheses about comets (p. 262); for some will have it that they are generated and perish as often as they appear and vanish; others, that they come from the regions of the fixed stars, and are seen by us in their passage through the system of our planets; and, lastly, others, that they are bodies perpetually revolving about the sun in very

eccentric orbits. In the first case, the comets, according to their different velocities, will move in conic sections of all sorts; in the second, they will describe hyperbolas, and in either of the two will frequent indifferently all quarters of the heavens; as well as those about the poles as those towards the ecliptic; in the third, their motions will be performed in ellipses very eccentric, and very nearly approaching to parabolas. But (if the law of the planets is observed) their orbits will not much decline from the plane of the ecliptic; and, so far as I could hitherto observe, the third case obtains; for the comets do, indeed, chiefly frequent the zodiac, and scarcely ever attain to a heliocentric latitude of  $40^{\circ}$ . And that they move in orbits very nearly parabolical, I infer from their velocity; for the velocity with which a parabola is described is every where to the velocity with which a comet or planet may be revolved about the sun in a circle at the same distance in the subduplicate ratio of 2 to 1 (by cor. 7, prop. 16); and, by my computation, the velocity of comets is found to be much about the same. I examined the thing by inferring nearly the velocities from the distances, and the distances both from the parallaxes and the phænomena of the tails, and never found the errors of excess or defect in the velocities greater than what might have arose from the errors in the distances collected after that manner. But I likewise made use of the reasoning that follows.

Supposing the radius of the *orbis magnus* to be divided into 1000 parts: let the numbers in the first column of the following table represent the distance of the vertex of the parabola from the sun's centre, expressed by those parts; and a comet in the times expressed in col. 2, will pass from its perihelion to the surface of the sphere which is described about the sun as a centre with the radius of the *orbis magnus*; and in the times expressed in col. 3, 4, and 5, it will double, triple, and quadruple, that its distance from the sun.

TABLE I.

The distance of a comet's perihelion from the sun's centre.	The radii of the <i>orbis magnus</i> .	The time of a comet's passage from its perihelion to a distance from the sun equal to					
		To its double.		To its triple.		To its quadruple.	
d. h. '	d. h. '	d. h. '	d. h. '	d. h. '	d. h. '	d. h. '	d. h. '
0	27 11 12	77 16 28	142 17 14	219 17 30			
5	27 16 07	77 23 14					
10	27 21 00	78 06 24					
20	28 06 40	78 20 13	144 03 19	221 08 54			
40	29 01 32	79 23 34					
80	30 13 25	82 04 56					
160	33 05 29	86 10 26	153 16 08	232 12 20			
320	37 13 46	93 23 38					
640	37 09 49	105 01 28					
1280		106 06 35	200 06 43	297 03 46			
2560			147 22 31	300 06 03			

[This table, here corrected, is made on the supposition that the earth's diurnal motion is just 59', and the measure of one minute loosely 0,2909, in respect of the radius 1000. If those measures are taken true, the true numbers of the table will all come out less. But the difference, even when greatest, and to the quadruple of the earth's distance from the sun, amounts only to 16<sup>h</sup>. 55'.]

The time of a comet's ingress into the sphere of the *orbis magnus*, or of its egress from the same, may be inferred nearly from its parallax, but with more expedition by the following

TABLE II.

The apparent elongation of a comet from the sun.	Its apparent diurnal motion in its own orbit.				Its distance from the earth in parts, whereof the radius of the <i>orbis magnus</i> contains 1000.
	Direct.		Retrog.		
60°	2° 18'	00°	20'		1000
65	2	33	00	35	845
70	2	55	00	57	684
72	3	07	01	09	618
74	3	23	01	25	551
76	3	43	01	45	484
78	4	10	02	12	416
80	4	57	02	49	347
82	5	45	03	47	278
84	7	18	05	20	209
86	10	27	08	19	140
88	18	37	16	39	70
90	Infinite		Infinite		00

The ingress of a comet into the sphere of the *orbis magnus*, or its egress from the same, happens at the time of its elongation from the sun, expressed in col. 1, against its diurnal motion. So in the comet of 1681, *Jan. 4*, O.S. the apparent diurnal motion in its orbit was about  $3^{\circ} 5'$ , and the corresponding elongation  $71\frac{2}{3}$ ; and the comet had acquired this elongation from the sun *Jan. 4*, about six in the evening. Again, in the year 1680, *Nov. 11*, the diurnal motion of the comet that then appeared was about  $4\frac{2}{3}^{\circ}$ ; and the corresponding elongation  $79\frac{2}{3}$  happened *Nov. 10*, a little before midnight. Now at the times named these comets had arrived at an equal distance from the sun with the earth, and the earth was then almost in its perihelion. But the first table is fitted to the earth's mean distance from the sun assumed of 1000 parts; and this distance is greater by such an excess of space as the earth might describe by its annual motion in one day's time, or the comet by its motion in 16 hours. To reduce the comet to this mean distance of 1000 parts, we add those 16 hours to the former time, and subduct them from the latter; and thus the former becomes *Jan. 4<sup>d</sup>. 10<sup>h</sup>. afternoon*; the latter *Nov. 10*, about six in the morning. But from the tenor and progress of the diurnal motions it appears that both

comets were in conjunction with the sun between *Dec. 7* and *Dec. 8*; and from thence to *Jan. 4<sup>d</sup>. 10<sup>h</sup>*. afternoon on one side, and to *Nov. 10<sup>d</sup>. 6<sup>h</sup>*. of the morning on the other, there are about 28 days. And so many days (by Table 1) the motions in parabolic trajectories do require.

But though we have hitherto considered those comets as two, yet, from the coincidence of their perihelions and agreement of their velocities, it is probable that in effect they were but one and the same; and if so, the orbit of this comet must have either been a parabola, or at least a conic section very little differing from a parabola, and at its vertex almost in contact with the surface of the sun. For (by Tab. 2) the distance of the comet from the earth, *Nov. 10*, was about 360 parts, and *Jan. 4*, about 630. From which distances, together with its longitudes and latitudes, we infer the distance of the places in which the comet was at those times to have been about 280: the half of which, viz. 140, is an ordinate to the comet's orbit, cutting off a portion of its axis nearly equal to the radius of the *orbis magnus*, that is, to 1000 parts. And, therefore, dividing the square of the ordinate 140 by 1000, the segment of the axis, we find the *latus rectum* 19,16, or in a round number 20; the fourth part whereof, 5, is the distance of the vertex of the orbit from the sun's centre. But the time corresponding to the distance of 5 parts in Tab. 1 is  $27^d. 16^h. 7'$ . In which time, if the comet moved in a parabolic orbit, it would have been carried from its perihelion to the surface of the sphere of the *orbis magnus* described with the radius 1000, and would have spent the double of that time, viz.  $55^d. 8\frac{1}{4}^h$ . in the whole course of its motion within that sphere: and so in fact it did; for from *Nov. 10<sup>d</sup>. 6<sup>h</sup>*. of the morning, the time of the comet's ingress into the sphere of the *orbis magnus*, to *Jan. 4<sup>d</sup>. 10<sup>h</sup>*. afternoon, the time of its egress from the same, there are  $55^d. 16^h$ . The small difference of  $7\frac{3}{4}^h$ . in this rude way of computing is to be neglected, and perhaps may arise from the comet's motion being some small matter flower, as it must have been if the true orbit in which it was carried was an ellipsis. The middle time between its

ingress and egress was *December 8<sup>d</sup>. 2<sup>h</sup>.* of the morning; and therefore at this time the comet ought to have been in its perihelion. And accordingly that very day, just before sun-rising, Dr. *Halley* (as we said) saw the tail short and broad, but very bright, rising perpendicularly from the horizon. From the position of the tail it is certain that the comet had then crossed over the ecliptic, and got into north latitude, and therefore had passed by its perihelion, which lay on the other side of the ecliptic, though it had not yet come into conjunction with the sun; and the comet [see more of this famous comet, p. 272 to 286, vol. 2] being at this time between its perihelion and its conjunction with the sun, must have been in its perihelion a few hours before; for in so near a distance from the sun it must have been carried with great velocity, and have apparently described almost half a degree every hour.

By like computations I find that the comet of 1618 entered the sphere of the *orbis magnus December 7*, towards sun-setting; but its conjunction with the sun was *Nov. 9, or 10*, about 28 days intervening, as in the preceding comet; for from the size of the tail of this, in which it was equal to the preceding, it is probable that this comet likewise did come almost into a contact with the sun. Four comets were seen that year, of which this was the last. The second, which made its first appearance *October 31*, in the neighbourhood of the rising sun, and was soon after hid under the sun's rays, I suspect to have been the same with the fourth, which emerged out of the sun's rays about *Nov. 9*. To these we may add the comet of 1607, which entered the sphere of the *orbis magnus Sep. 14*, O.S. and arrived at its perihelion distance from the sun about *October 19*, 35 days intervening. Its perihelion distance subtended an apparent angle at the earth of about 23 degrees, and was therefore of 390 parts. And to this number of parts about 34 days correspond in Tab. 1. Farther; the comet of 1665 entered the sphere of the *orbis magnus* about *March 17*, and came to its perihelion about *April 16*, 30 days intervening. Its perihelion distance subtended an angle at the earth of about seven degrees, and therefore was of 122 parts:

and corresponding to this number of parts, in Tab. 1, we find 30 days. Again; the comet of 1682 entered the sphere of the *orbis magnus* about Aug. 11, and arrived at its perihelion about Sep. 16, being then distant from the sun by about 350 parts, to which, in Tab. 1, belong  $33\frac{1}{2}$  days. Lastly; that memorable comet of *Regiomontanus*, which in 1472 was carried through the circum-polar parts of our northern hemisphere with such rapidity as to describe 40 degrees in one day, entered the sphere of the *orbis magnus* Jan. 21, about the time that it was passing by the pole, and, hastening from thence towards the sun, was hid under the sun's rays about the end of Feb.; whence it is probable that 30 days, or a few more, were spent between its ingress into the sphere of the *orbis magnus* and its perihelion. Nor did this comet truly move with more velocity than other comets, but owed the greatness of its apparent velocity to its passing by the earth at a near distance.

It appears, then, that the velocity of comets (p. 268), so far as it can be determined by these rude ways of computing, is that very velocity with which parabolas, or ellipses near to parabolas, ought to be described; and therefore the distance between a comet and the sun being given, the velocity of the comet is nearly given. And hence arises this problem.

### PROBLEM.

*The relation betwixt the velocity of a comet and its distance from the sun's centre being given, the comet's trajectory is required.*

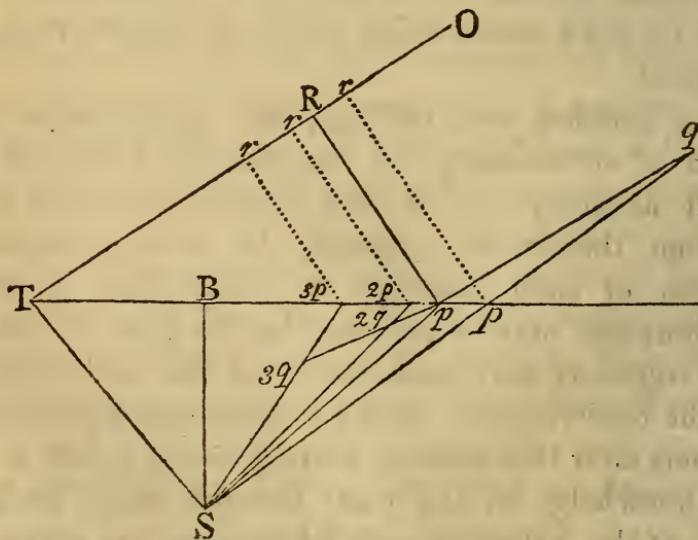
If this problem was resolved, we should thence have a method of determining the trajectories of comets to the greatest accuracy; for if that relation be twice assumed, and from thence the trajectory be twice computed, and the error of each trajectory be found from observations, the assumption may be corrected by the Rule of False, and a third trajectory may thence be found that will exactly agree with the observations. And by determining the trajectories of comets after this method, we may come, at last, to a more exact knowledge of the parts through which those bodies travel, of the velocities with which they are carried, what sort of trajectories they describe, and what are the true mag-

nitudes and forms of their tails according to the various distances of their heads from the sun; whether, after certain intervals of time, the same comets do return again, and in what periods they complete their several revolutions. But the problem may be resolved by determining, first, the hourly motion of a comet to a given time from three or more observations, and then deriving the trajectory from this motion. And thus the invention of the trajectory, depending on one observation, and its hourly motion at the time of this observation, will either confirm or disprove itself; for the conclusion that is drawn from the motion only of an hour or two and a false hypothesis will never agree with the motions of the comets from beginning to end: The method of the whole computation is this.

## LEMMA I.

To cut two right lines OR, TP, given in position, by a third right line RP, so as TRP may be a right angle; and, if another right line SP is drawn to any given point S, the solid contained under this line SP, and the square of the right line OR terminated at a given point O; may be of a given magnitude.

It is done by linear description thus. Let the given magnitude of the solid be  $M^2 \times N$ ; from any point  $r$  of the right line  $OR$  erect the perpendicular  $rp$  meeting  $TP$  in  $p$ . Then



through the point Sp draw the line Sq equal to  $\frac{M^2 \times N}{Or^2}$ . In like manner draw three or more right lines S2q, S3q, &c.; and a regular line q2q3q drawn through all the points q2q3q, &c. will cut the right line TP in the point P, from which the perpendicular PR is to be let fall. Q.E.F.

By trigonometry thus. Assuming the right line TP as found by the preceding method, the perpendiculars TR, SB, in the triangles TPR, TPS, will be thence given; and the side SP in the triangle SBP, as well as the error  $\frac{M^2 \times N}{Or^2} - SP$ . Let this error, suppose D, be to a new error, suppose E, as the error  $2p_2q \pm 3p_3q$  to the error  $2p_3p$ ; or as the error  $2p_2q \pm D$  to the error  $2p_P$ ; and this new error added to or subducted from the length TP, will give the correct length  $TP \pm E$ . The inspection of the figure will shew whether we are to add or to subtract; and if at any time there should be use for a farther correction, the operation may be repeated.

By arithmetic thus. Let us suppose the thing done, and let  $TP + e$  be the correct length of the right line TP as found out by delineation; and thence the correct lengths of the lines OR, BP, and SP, will be  $OR - \frac{TR}{TP}e$ ,  $BP + e$ , and  $\sqrt{SP^2 + 2BPe + ee} = \frac{M^2N}{OR^2 + \frac{2OR+TR}{TP}e + \frac{TR^2}{TP^2}ee}$ .

Whence, by the method of converging series, we have  $SP + BP - \frac{SB^2}{2SP^3}ee + \&c. = \frac{M^2N}{OR^2} + \frac{2TR}{TP} \times \frac{M^2N}{OR^3}e + \frac{3TR^2}{TP^2} \times \frac{M^2N}{OR^4}ee + \&c.$  For the given co-efficients  $\frac{M^2N}{OR^2} - SP$ ,  $\frac{2TR}{TP}$

$\times \frac{M^2N}{OR^3} - \frac{BP}{SP}$ ,  $\frac{3TR^2}{TP^2} \times \frac{M^2N}{OR^4} - \frac{SB^2}{2SP^3}$ , putting  $F, \frac{F}{G}, \frac{F}{GH}$ , and carefully observing the signs, we find  $F + \frac{F}{G}e + \frac{F}{GH}ee = 0$ , and  $e + \frac{ee}{H} = -G$ . Whence, neglecting the very

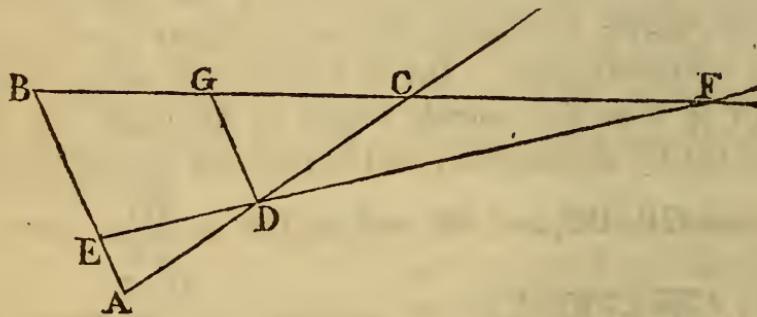
small term  $\frac{e^2}{H}$ ,  $e$  comes out equal to  $-G$ . If the error  $\frac{e^2}{H}$  is not despicable, take  $-G - \frac{G^2}{H} = e$ .

And it is to be observed that here a general method is hinted at for solving the more intricate sort of problems, as well by trigonometry as by arithmetic, without those perplexed computations and resolutions of affected equations which hitherto have been in use.

### LEMMA II.

*To cut three right lines given in position by a fourth right line that shall pass through a point assigned in any of the three, and so as its intercepted parts shall be in a given ratio one to the other.*

Let  $AB$ ,  $AC$ ,  $BC$ , be the right lines given in position, and suppose  $D$  to be the given point in the line  $AC$ . Parallel to



$AB$  draw  $DG$  meeting  $BC$  in  $G$ ; and, taking  $GF$  to  $BG$  in the given ratio, draw  $FDE$ ; and  $FD$  will be to  $DE$  as  $FG$  to  $BG$ . Q.E.E.

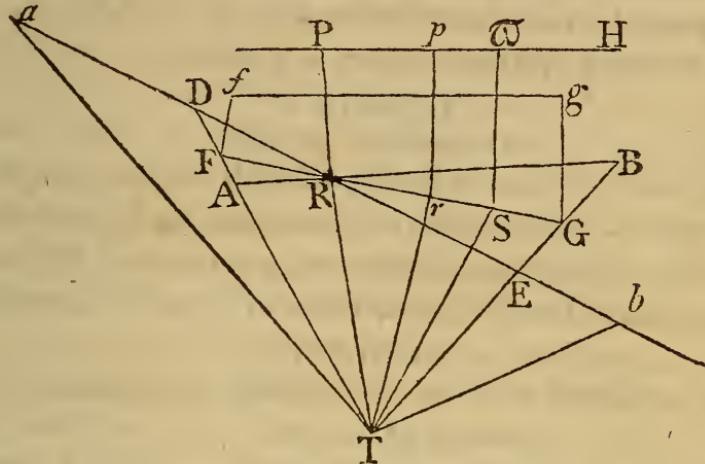
By trigonometry thus. In the triangle  $CGD$  all the angles and the side  $CD$  are given, and from thence its remaining sides are found; and from the given ratios the lines  $GF$  and  $BE$  are also given.

### LEMMA III.

*To find and represent by a linear description the hourly motion of a comet to any given time.*

From observations of the best credit, let three longitudes of the comet be given, and, supposing  $ATR$ ,  $RTB$ , to be their

differences, let the hourly motion be required to the time of the middle observation TR. By *Lem. II.* draw the right line ARB, so as its intercepted parts AR, RB, may be as the times between the observations; and if we suppose a body in the whole time to describe the whole line AB with an equal



motion, and to be in the mean time viewed from the place T, the apparent motion of that body about the point R will be nearly the same with that of the comet at the time of the observation TR.

*The same more accurately.*

Let Ta, Tb, be two longitudes given at a greater distance on one side and on the other; and, by *Lem. II.* draw the right line aRb so as its intercepted parts aR, Rb, may be as the times between the observations aTR, RTb. Suppose this to cut the lines TA, TB, in D and E; and because the error of the inclination TRa increases nearly in the duplicate ratio of the time between the observations, draw FRG, so as either the angle DRF may be to the angle ARE, or the line DF to the line AF, in the duplicate ratio of the whole time between the observations aTB to the whole time between the observations ATB, and use the line thus found FG in place of the line AB found above.

It will be convenient that the angles ATR, RTB, aTA, BTb, be no less than of ten or fifteen degrees, the times corresponding no greater than of eight or twelve days, and the

longitudes taken when the comet moves with the greatest velocity; for thus the errors of the observations will bear a less proportion to the differences of the longitudes.

#### LEMMA IV.

*To find the longitudes of a comet to any given times.*

It is done by taking in the line FG the distances  $Rr$ ,  $R\varrho$ , proportional to the times, and drawing the lines  $Tr$ ,  $T\varrho$ . The way of working by trigonometry is manifest.

#### LEMMA V.

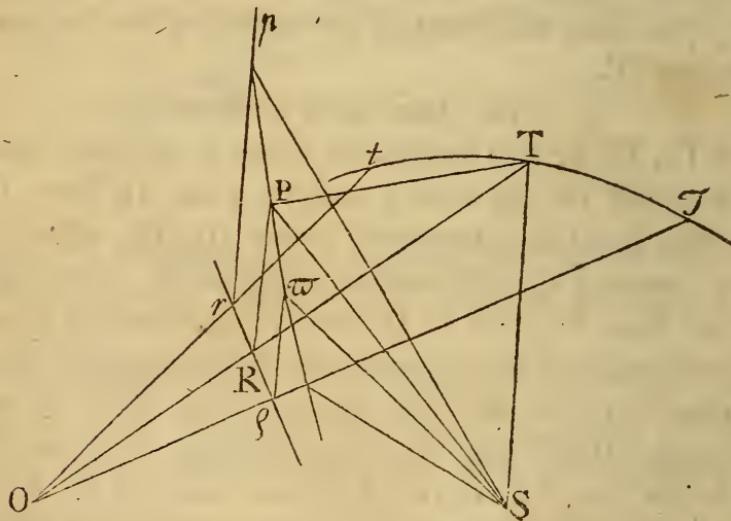
*To find the latitudes.*

On  $TF$ ,  $TR$ ,  $TG$ , as radii, at right angles erect  $Ff$ ,  $RP$ ,  $Gg$ , tangents of the observed latitudes; and parallel to  $fg$  draw  $PH$ . The perpendiculars  $rp$ ,  $\varrho\varpi$ , meeting  $PH$ , will be the tangents of the sought latitudes to  $Tr$  and  $T\varrho$  as radiuses.

#### PROBLEM I.

*From the assumed ratio of the velocity to determine the trajectory of a comet.*

Let  $S$  represent the sun;  $t$ ,  $T$ ,  $\tau$ , three places of the earth in its orbit at equal distances;  $p$ ,  $P$ ,  $\varpi$ , as many corresponding



places of the comet in its trajectory, so as the distances interposed betwixt place and place may answer to the motion of one hour;  $pr$ ,  $PR$ ,  $\varpi\varrho$ , perpendiculars let fall on the plane of the ecliptic, and  $rR\varrho$  the vestige of the trajectory in this plane. Join  $Sp$ ,  $SP$ ,  $S\varpi$ ,  $SR$ ,  $ST$ ,  $tr$ ,  $TR$ ,  $\tau\varrho$ ,  $TP$ ; and let  $tr$ ,  $\tau\varrho$ , meet in

O, TR will nearly converge to the same point O, or the error will be inconsiderable. By the premised lemmas the angles rOR, RO $\rho$ , are given, as well as the ratios pr to tr, PR to TR, and  $\varpi\rho$  to  $\tau\rho$ . The figure tT $\tau$ O is likewise given both in magnitude and position, together with the distance ST, and the angles STR, PTR, STP. Let us assume the velocity of the comet in the place P to be to the velocity of a planet revolved about the sun in a circle, at the same distance SP, as V to 1; and we shall have a line pP $\varpi$  to be determined, of this condition, that the space p $\varpi$ , described by the comet in two hours, may be to the space V  $\times$  t $\tau$  (that is, to the space which the earth describes in the same time multiplied by the number V) in the subduplicate ratio of ST, the distance of the earth from the sun, to SP, the distance of the comet from the sun; and that the space pP, described by the comet in the first hour, may be to the space P $\varpi$ , described by the comet in the second hour, as the velocity in p to the velocity in P; that is, in the subduplicate ratio of the distance SP to the distance Sp, or in the ratio of 2Sp to SP + Sp; for in this whole work I neglect small fractions that can produce no sensible error.

In the first place, then, as mathematicians, in the resolution of affected equations, are wont, for the first essay, to assume the root by conjecture, so, in this analytical operation, I judge of the sought distance TR as I best can by conjecture. Then, by Lem. II. I draw r $\rho$ , first supposing rR equal to R $\rho$ , and again (after the ratio of SP to Sp is discovered) so as rR may be to R $\rho$  as 2SP to SP + Sp, and I find the ratios of the lines p $\varpi$ , r $\rho$ , and OR, one to the other. Let M be to V  $\times$  t $\tau$  as OR to p $\varpi$ ; and because the square of p $\varpi$  is to the square of V  $\times$  t $\tau$  as ST to SP, we shall have, *ex aequo*, OR $^2$  to M $^2$  as ST to SP, and therefore the solid OR $^2$   $\times$  SP equal to the given solid M $^2$   $\times$  ST; whence (supposing the triangles STP, PTR, to be now placed in the same plane) TR, TP, SP, PR, will be given, by Lem. I. All this I do, first by delineation in a rude and hasty way; then by a new delineation with greater care; and, lastly, by an arithmetical computation. Then I proceed to determine the position of the lines r $\rho$ , p $\varpi$ , with the greatest accuracy, together with the nodes and incli-

nation of the plane  $S p \varpi$  to the plane of the ecliptic ; and in that plane  $S p \varpi$  I describe the trajectory in which a body let go from the place P in the direction of the given right line  $p \varpi$  would be carried with a velocity that is to the velocity of the earth as  $p \varpi$  to  $V \times t \tau$ . Q.E.F.

### PROBLEM II.

*To correct the assumed ratio of the velocity and the trajectory thence found.*

Take an observation of the comet about the end of its appearance, or any other observation at a very great distance from the observations used before, and find the intersection of a right line drawn to the comet, in that observation with the plane  $S p \varpi$ , as well as the comet's place in its trajectory to the time of the observation. If that intersection happens in this place, it is a proof that the trajectory was rightly determined ; if otherwise, a new number V is to be assumed, and a new trajectory to be found ; and then the place of the comet in this trajectory to the time of that probatory observation, and the intersection of a right line drawn to the comet with the plane of the trajectory, are to be determined as before ; and by comparing the variation of the error with the variation of the other quantities, we may conclude, by the Rule of Three, how far those other quantities ought to be varied or corrected, so as the error may become as small as possible. And by means of these corrections we may have the trajectory exactly, providing the observations upon which the computation was founded were exact, and that we did not err much in the assumption of the quantity V ; for if we did, the operation is to be repeated till the trajectory is exactly enough determined. Q.E.F.

*End of the System of the World.*

A  
**SHORT COMMENT**  
ON  
SIR ISAAC NEWTON'S  
**PRINCIPIA.**  
CONTAINING  
**EXPLANATIONS OF SOME DIFFICULT PLACES**  
IN  
THAT EXCELLENT WORK.  
~~~~~

*By W. Emerson.*



## PREFACE.

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THE PRINCIPIA being a book which is universally read by all the world, that pretend to any degree of philosophical learning, it cannot be improper to explain such passages therein as seem obscure and difficult. For although it is written in as clear a style as can be done in so few words, yet, by reason of its conciseness, and the difficulty of the subjects treated on, many things occur which require some farther explication, especially to young beginners.

Accordingly, several mathematical writers have endeavoured to explain some parts or other of this Work, to make them intelligible to common readers, who, without such helps, would find it very difficult to get forward.

The noble subjects this work treats of being no less than the grand fabric of the world, and the whole system of nature, although comprised in so little a compass, makes it highly deserving of every illustration that can be given it.

The author has clearly shewn in this work that all the bodies in the world are actuated by the universal principle of gravity ; which is this, that every body is attracted or impelled towards any other body by a force which is reciprocally as the square of the distance of the two bodies. On this universal principle he shews that the motions of all the great bodies in the world are founded.

Several men had written systems of Philosophy before Sir *Isaac* ; but, from their ignorance of nature, none of them could stand the test. But his Principles being built upon the unerring foundation of observations and experiments, must necessarily stand good, till the dissolution of nature itself.

This little Treatise was written many years since ; for when I studied the Principia, I was frequently at a stop, which obliged me to make calculations here and there as I went on ; and, when I had done, I set them down as notes upon these places ; wherein I only meddled with those places that appeared difficult to me. These notes collected together are the subject of the following Comment ; and I have revised the whole, and added several things that seemed

wanting. Yet I believe there are some things still behind, which are not sufficiently explained by any commentator, and especially such as are there laid down without their demonstrations.

As nobody is reckoned a philosopher that does not read the Principia, therefore I thought proper to publish this small Tract, supposing that it may be useful to others that have a desire to read that celebrated Work. What I have farther to mention is this: the passages referred to, and which are to be explained, are not taken from the Latin edition, which would not suit the English reader; but from Motte's translation, and from the first edition thereof, if there happen to be more.

*W. Emerson.*

## ADVERTISEMENT.

Besides the common Algebraic characters which are in use, I make use of this  $\alpha$ , which signifies a proportion. Thus  $A \propto BC$ , signifies that A is in a constant ratio to BC, or that A is as BC.

## SHORT COMMENT,

&c.

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[COR. 2, to the laws. And therefore if the weight p is to the weight A, &c.] For p will have the same effect as P, if  $p : P :: pH \times OL : pN \times$  perpendicular from O on pN. And  $P : A :: (by\ what\ went\ before) KO : OL$ . Therefore, *ex equo*, if p and A are in equilibrio, it will be  $p : A :: pH \times KO : pN \times$  perpendicular from O on pN. Note, the line pN ought to be drawn in fig. 2, and not PL.

[Cor. 4, ib.] This is demonstrated in Keil's Introduction, theor. 20; as likewise in lem. 23, p. 86, of this Work.

[Ib. p. 20, *sub finem*; but the distance between these two centres,] that is, between the centre of the two, and the centre of all the rest. Farther, the actions of all the bodies may be considered as the sum of the actions of every two; and then the case will be plain.

[Sch. to cor. 6, p. 23, then will ST represent] For if  $RV =$  retardation of describing  $2RA + 2VA$ , ST will be the retardation of describing  $\frac{1}{4}$  of that, or  $\frac{1}{2}RV + \frac{1}{2}VA$ , that is, SA or TA; and therefore the body falling from S in the air, or from T *in vacuo*, will have, nearly, the same velocity in A; the same of sA or tA; for ascending to s in the air, or ascending to t (or descending from t to A) *in vacuo* has the same velocity in A.

[Ib. p. 27, but if they are turned aside by the interposition.] This is plain by prop. 16, Mechanics.

[Ib. And in like manner, &c.] By the same prop. as before.

## BOOK I.

## SECTION I.

[Lem. 10.] For let AE be divided into an infinite number of equal parts, any part, as D',  $\times$  by the velocity acquired in the time AD (that is,  $D \times DB$ ), is as the space described in that little part of time D' (for the spaces are as the times  $\times$  by the velocities); and the sum of all these products or areas, that is, the areas ADB, AEC, are as the whole spaces described in the times AD, AE; but these areas are as  $AD^2$  and  $AE^2$ , by lem. 9.

[Ib. cor. 1. — to the bodies, and measured — ] that is, and the said errors measured by the distances of the bodies, &c.

[Lem. 11, have a finite curvature.] These words exclude those curves whose radius of curvature is infinitely small, or infinitely great.

[Ib. schol.] All this may be universally demonstrated after this manner. Let AB, AF (Fig. 1), be two paraboloids. Let *latus rectum* of AF = a, of AB = b, AC = z, AE = x, CB, EF, or AD = y. And let m be any affirmative index, and suppose  $a^m x = y^m + ^1$ . And  $b^m + ^n z = y^m + ^n + ^1$ . Then  $\frac{^m}{a^m + ^1} \frac{^1}{x^m + ^1} = y = \frac{^m + ^n}{b^m + ^n + ^1} \times \frac{^1}{z^m + ^n + ^1}$ . And by involution,  $a^m \times \frac{^m + ^n + ^1}{z^m + ^n + ^1} x^m + ^n + ^1 = \frac{^m + ^n}{b^m + ^n} \times \frac{^m + ^1}{z^m + ^1} z^m + ^1$ . And, therefore,  $a^{m^2} + ^{mn} + ^m x^m : b^{m^2} + ^{mn} + ^m + ^n : z^m + ^1 : x^m + ^1$ . But, because x is infinitely small, therefore when n is affirmative, the first term is infinitely less than the second; and, therefore,  $z^m + ^1$  is infinitely lesser than  $x^m + ^1$ ; but if n is negative, the first term is infinitely greater than the second (Fig. 1), and, therefore,  $z^m + ^1$  is infinitely greater than  $x^m + ^1$ , or z than x; that is, in case the first DB is infinitely less, and in case the second infinitely greater than DF. But DB, DF, are as the curvatures of AB, AF, therefore, &c.

## SECTION II.

[Pr. 1, cor. 4, as the versed sines, of arcs] (Fig. 2) described in equal times, tending to the centre of force, and

bisect the chords; let  $AB = BC$ , and  $Bd$  be  $\perp$  to  $AC$ , then  $Ad = dC$ . And when the arc  $AC$  is diminished to infinity,  $e$  coincides with  $d$ ; and consequently  $Ae = eC$ , or  $Be$  bisects the chord  $AC$ . Complete the parallelogram  $ABCf$ , and  $eB = \frac{1}{2}fb$ . Note.—He calls that the versed sine of an arc which is commonly called the versed sine of half that arc.

[Pr. 4, cor. 2.] For the forces  $\propto \frac{\text{vel.}^2}{\text{rad.}} \propto \frac{\text{vel.}^2 \times \text{rad.}}{\text{rad.}} \propto \frac{\text{rad.}}{p \text{ times}^2}$ ; all the other corollaries (except the last) depend on this.

[Ib. cor 8.] Let  $R$  = radius of curvature,  $D$  = distance from the centre of force. Then will the p. time  $\propto R^n \propto D^n$  (by similar position), and the areas  $\propto$  velocities  $\propto R^{n-1} \propto D^{n-1}$ . And therefore the force  $\propto \frac{1}{D^{2n-1}}$ ; and the contrary.

[Ibid, cor. 9.] Take the arc  $Bd$  infinitely small, and let  $BF$  be described by the revolving body (Fig. 3). in the same time that a body falls from  $B$  to  $E$  by an uniform centripetal force which it has at  $B$ ; then  $cd$  is the space fallen in the time of describing  $Bd$ . But  $Cd$  or  $Bn : BE ::$  (as the squares of the times, that is, as)  $Bd^2 : BF^2 :: \frac{Bd^2}{BA} : \frac{BF^2}{BA}$ . But  $Bn = \frac{Bd^2}{BA}$ . Therefore  $BE = \frac{BF^2}{BA}$ . And  $BE : BF : BA ::$ .

[Ib. schol. as the square of the length applied to the radius ;] for the number of reflections is  $\propto$  velocity or length directly, and the radius reciprocally.

[ Pr. 7, cor. 2, in the same periodical time] Let  $ac$  be  $\parallel$  to  $RP$ , and  $da \parallel$  to  $PS$  (Fig. 4). Then if the periodic times be equal, the areas generated in a given infinitely small time must be equal, that is, the velocities round  $R$  and  $S$  must be reciprocally as (the  $\perp$ s on  $PG$  from  $R$  and  $S$ , that is, as)  $RP$  and  $SG$ . And supposing  $a, c, d$ , to coincide in  $P$ , the force round  $R$  to the force round  $S$  is in the complicate ratio of (of  $ac$  to  $ad$ , or)  $SG$  to  $SP$ , and the squares of the times of describing a given arc, that is reciprocally as the squares

of the velocities, that is, as  $SG^2$  to  $RP^2$ . Therefore the force round R to the force round S :: is as  $SG^3$  to  $SP \times RP^2$ , when the periodical times are equal.

Or thus : let p be the place of the body when the tangent pg is  $\parallel$  to the line RS. Then the velocities round R and S, in the place p, will be equal ; for the small areas are equal, and their heights are equal, by reason of the parallels RS, pg. Draw TV, tv, and then by similar triangles  $Sp^3 \times Pv^3 = Sg^3 \times pt^3$ . And  $Sp^3 \times Pv^3 = SG^3 \times PT^3$ . Then by this prop. force round S, in P : force round S, in p ::  $Sp^2 \times Pv^3$  or  $\frac{Sg^3 \times pt^3}{Sp}$  :  $SP^2 \times PV^3$  or  $\frac{SG^3 \times PT^3}{SP}$ .

And force round S, in p : force round R, in p ::  $Sp : Rp$  or  $Sg$ .

Also force round R, in p : force round R, in P ::  $RP^2 \times PT^3 : Rp^2 \times pt^3$  or  $Sg^2 \times pt^3$  :: ex equo, force round S, in P : force round R, in P ::  $RP^2 \times PT^3 : \frac{SG^3 \times PT^3}{SP} :: SP \times RP^2 : SG^3$ .

[Cor. 3, in the same periodic time] for then the infinitely small and equal areas will be described in equal times in P, and both these areas and the forces will be the same as in a circle of the same curvature with the orbit at P, and therefore the forces are the same as in the foregoing corol.

[Pr. 8, sch.] let ApD (Fig. 5) be an ellipsis, AP a circle. Then  $rn : Rn :: (pm : Pm) : qn : Qu$ . And by division,  $qr : QR :: (qn : Qn) : CD : CA$ . The force in the ellipsis  $\propto \frac{qt^2 \times Sp^2}{qr}$  reciprocally = (because  $qr = \frac{QR \times CD}{CA}$ )  $\frac{QT^2 \times CA \times Sp^2}{QR \times CD} =$  (because  $\frac{QT^2}{QR} = \frac{2Pm^3}{CA^2} \frac{2Pm^3 \times Sp^2}{CA \times CD}$ )  $=$  (because  $Pm^3 = \frac{pm^3 \times CA^3}{CD^3}$ )  $\frac{2pm^3 \times CA^2 \times Sp^2}{CD^4}$ . Therefore (because CA, Sp, and CD are given) the force in the ellipsis is  $\propto pm^3$  reciprocally. But in the hyperbola and parabola (where CA is negative or infinite) these lines are still given, and therefore the force in any conic section is reciprocally as  $mp^3$ .

[Pr. 9, will be changed —] that is  $\frac{QT^2}{QR}$  is every where the same ratio, viz.  $\propto SP$ .

[Ib. second way]. For PV is (by reason of the given angle at P) as the radius of curvature, that is (by reason of the similarity of the parts of the figure PQ), as SP.

[Prop. 10, second way. Add the rectangle uPv] for  $Qv^2 + uPv = Qv^2 + \overline{TP} + \overline{Tv} \times \overline{TP} - \overline{Tv} = Qv^2 + TP^2 - Tv^2 = QT^2 + TP^2 =$  square of the chord QP; and  $Pv \times uV + uPv = VPv$ .

And if the line QV be drawn, and a circle through the points PQV; the triangles PQv and PQV will be similar (the  $\angle QVP$  being  $= QPR = PQv$ ), and therefore  $Pv : PQ : PV ::$ .

[Prop. 10, sch. in the ratio of the distances from the centre] for the fluxion of the ordinates is augmented or diminished in the same ratio, and that is as the force.

### SECTION III.

[Prop. 13, cor. 2]. For, in the demonstrations of prop. 11, 12, and 13,  $QT^2$  is always equal to  $QR \times latus rectum$ .

[Prop. 16, cor.] The four first corollaries are general, and agree to all conic sections; the sixth corollary belongs to one and the same parabola.

[Ib. cor. 6, it is more variable] that is, in the ellipsis, the ratio of the velocity at a less and greater distance is greater than the ratio of the square roots of the greater and lesser distance: in the hyperbola it is greater; for the ratio of the greater and lesser perpendiculars (which is the same with this ratio of the velocities) is greater than the ratio of the square roots of the greater and lesser distances in the ellipsis, and lesser than it in the hyperbola; for in the ellipsis the perpendiculars in the greatest and least distances are the same with these distances; and in the hyperbola the greatest perpendicular possible is that from the focus on the asymptote, and the least the distance to the vertex; also (by Conics, prop. 24, ellipsis) the perpendicular SY  $\propto \sqrt{\frac{SP}{HP}}$  (Fig. 3, see also cor. 7).

[Prop. 17, yet greater velocity] then PH (and PK) will be negative.

[Ib. cor. 2] (Fig. 5) The velocity in a circle is = velocity acquired by falling through  $\frac{1}{2}DS$ , by the given centripetal force.

## SECTION IV.

[Prop. 18, 19, 20.] Two given points or right lines.

[Prop. 20, case 4. But because of the similar triangles VSH, vsh] ; for (by similar  $\Delta$ s, SVP, shq)  $SV : SP :: sh : sq$  ; (by similar  $\Delta$ s, svh, spq)  $sv : sp$ . And by the similar  $\Delta$ s SHP, shp)  $SP : SH :: sp : sh$  ; ergo  $SV : SH :: sv : sh$  ; and the angle  $VSH = psq = vsh$ .

[Prop. 21] three—lines. If three tangents be given, you have three points Y, from which three equal right lines as YH are to be drawn to H, by case 3 of the last lemma. If two tangents and a point P, there will be given two points Y, from which two equal lines are to be drawn to the focus H, and a third point P, from which PH is to be drawn ; by case 2, lem. In the hyperbola, it is  $PH - YH = SP$ . If three points P be given, it is done by case 1, lem. 16.

## SECTION V.

[Lem. 17, in given angles] each to each respectively in the same invariable angle, though they are not all equal.

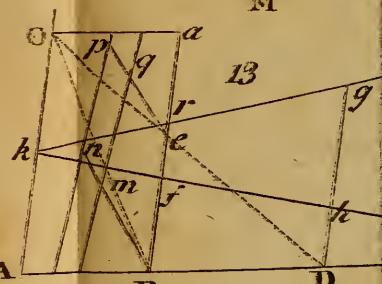
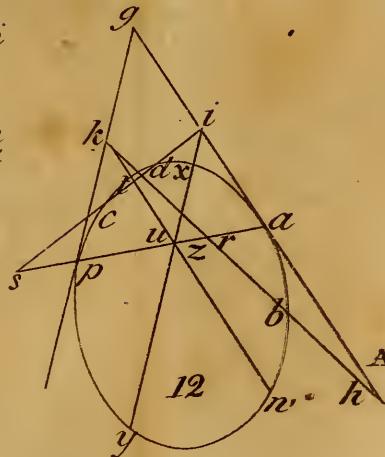
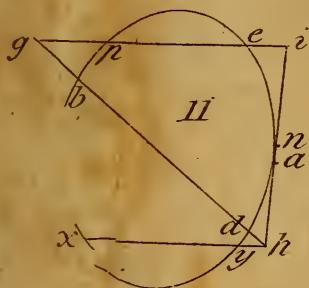
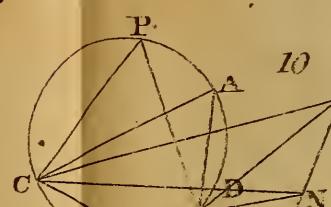
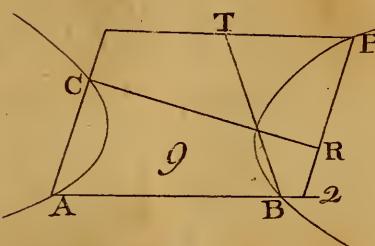
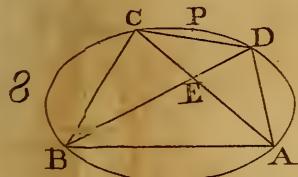
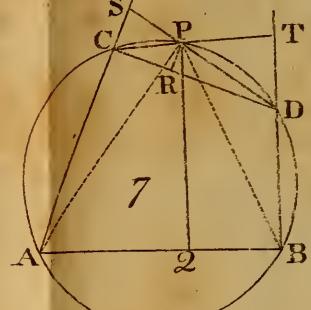
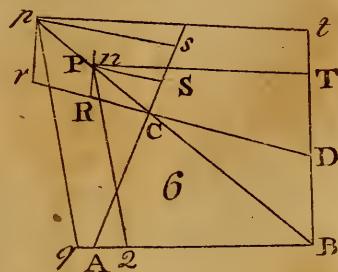
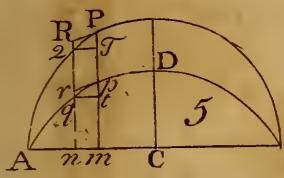
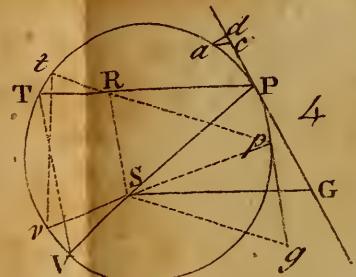
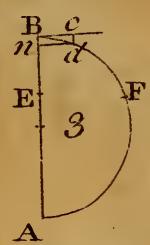
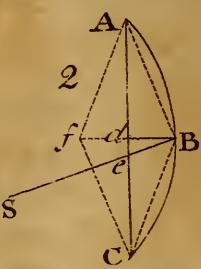
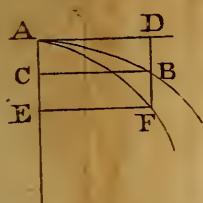
[Lem. 18, and so (by supposition)—] for p and b are in the curve, and the rectangles of the lines from p and P are in the given ratio.

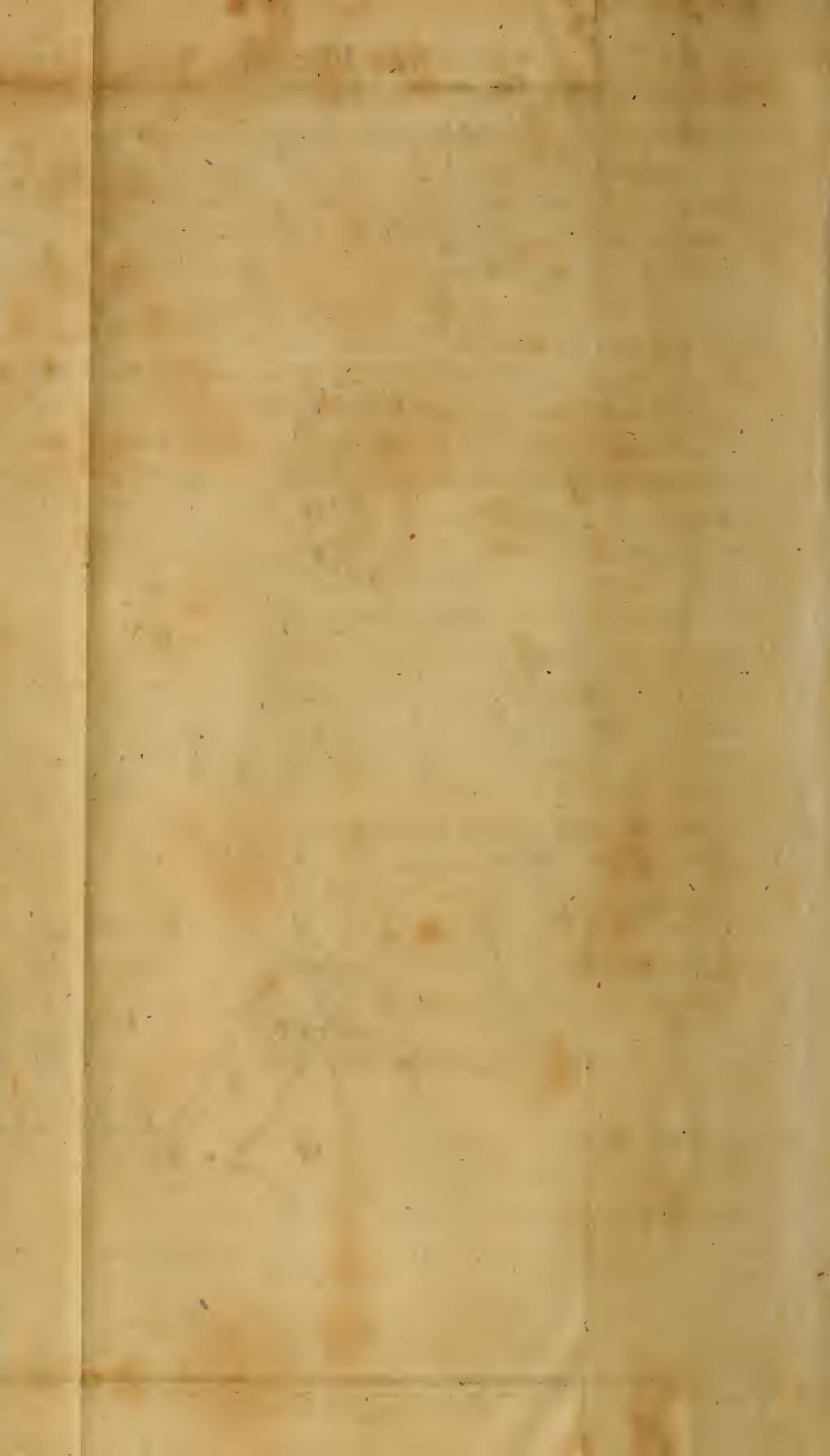
[Ib. sch. if p happens to be in a right line] For if  $rpq : spt :: RPQ : SPT$ ; and P be placed in pB; then by similar triangles  $pq : PQ :: (pB : PB ::) pt : PT$  (Fig. 6); and  $rp : RP :: (Cp : CP ::) ps : PS$ . And by multiplying you have  $rpq : RPQ :: spt : SPT$ ; which agrees with the lemma. But if P is placed out of pB, as at n; then because the ratio  $RnQ$  is greater than  $\frac{RPQ}{SPT}$ , it is also greater than  $\frac{rpq}{spt}$ , which SnT is against the hypothesis (Fig. 6).

If p is placed in DC, then  $rpq$  will be  $= 0$ , and therefore  $RPQ = 0$ ; and P will also be placed in DC.

[Ib. If the two opposite angles] (Fig. 7) The  $\Delta$ s PCR, PBT are similar, for the angles at S and T are right, and C  $= B$ , being on the same arc PD; therefore  $PR : PC :: PT : PB$ ; and the triangles PBQ and SCP are similar, for the angles at S, Q are right, and  $SCP = PBA$ ; therefore  $PC : PS :: PB : PQ$ ; and, ex equo,  $PR \times PQ = PS \times PT$ ;

Fig. 1.





and, *e contra*, if  $PR \times PQ = PS \times PT$ , the locus of the point P is a circle.

And if these lines are not perpendicular, yet, since their lengths will be reciprocally as the sines of the angles, it will follow that  $PQ \times PR : PS \times PI :: \sin S \times \sin T : \sin Q \times \sin R$ , and the contrary.

[Ib. and one or two] (Fig. 8) If the point B be supposed to move towards A, and at last to coincide with it, so that AB become a tangent at A, the lemma will still hold. And if B pass beyond A, then the figure will be converted into this BECDEAB; the lemma will still hold as before: and if B move to an infinite distance, then DB, AB will be parallels; and C also, then DC, AC will also be parallel; and also the conic section passing through A and D will pass on infinitely towards C and B.

[Lem. 19.] This may be resolved as prob. 12, in the Universal Arithmetic.

[Lem. 20.] Here is another figure relating to this lemma (Fig. 9).

[Lem. 21. Therefore (by lem. 20) the point D] for the angles CPR, BPT, CPB are given, and therefore the lines PT, PR (to which the sides AQ, AS of the parallelogram are parallel) are given by position (See fig. above).

Here is another figure of lem. 21 (Fig. 10). This lemma is the same with prob. 53, of the Universal Arithmetic.

[Prop. 22.] (Fig. 11) This is the same with prob. 55, in the author's Universal Arithmetic.

[Prop. 23, case 1.] This is the same with prob. 56, of the Universal Arithmetic, or prob. 57.

[Prop. 23 (Fig. 11), case 2.  $HA^2$  will become to  $AI^2$ ] for let n be a point in the conic section infinitely near a, through both which the line ih passes; then ih is a tangent at a. Then (cor. 4, prop. 44, b. I. my Conics) it will be,  $bgd : bhd :: pgc : xhy = \frac{bhd \times pgc}{bgd}$ , and  $hna$ , or  $ha^2 : ian$ , or  $ia^2 : xhy$ , or  $\frac{bhd \times pgc}{bgd} : pic :: bhd \times pgc : pic \times bgd$ .

[Prop. 24, — will (by the properties of the conic sections)] (Fig. 12) because any tangent may be supposed to cut the curve in two points infinitely near each other, therefore

(by cor. 4, prop. 44, b. I. Ellipsis) these proportions follow. Also draw  $kw \parallel$  to  $ga$ , to intersect  $pg$ , and let  $nk^2 = tkw$ . Then  $tkw$  or  $nk^2 : ha^2 :: dkb : bhd :: kr^2 : hr^2$ . And  $nk : ha :: kr : hr$ . And  $n, r, a$ , fall in one right line. Again,  $kn : ga :: kp^2 : gp^2$ ; and  $kn : gn :: kp : gp$ . Whence  $n, p, a$  fall in one right line. And therefore the points  $p, r, a$ , are in one right line.

[Lem. 22. Thus any right lines converging] Let the lines be  $kg, kh$  (Fig. 13). Draw  $OB, OeD$ , and  $pe \parallel$  to  $nB$ . Let  $m, q$ , be the projected points of  $f$  and  $h$ . Then (because in the point  $B$ ,  $OB = OD$ )  $mn$  (or  $Bn - Bm$ )  $= fr$ . Also  $qp$  (or  $ep - eq$ )  $: hg$  (or  $Dg - Dh$ )  $:: Oe : OD :: AB : AD :: fr$ , or  $mn : hg :: Oe : OD :: AB : AD :: fr$ . Therefore  $qp = mn$ , therefore  $pn, qm$  (which are the projected lines of  $gr, hf$ ), are parallel.

[Ib. we shall have the solution required]. (Fig. 13) For the figure  $hgi$  supposed now to be given, may be transformed into the first figure  $HGI$ , by making as  $Od$  to  $dg$ , so is  $OD$  to  $DG$  parallel to the radius  $AO$ .

[Ib. For as often as two conic sections] For these conic sections being transformed into simpler ones, give the point of intersection; and thereby is had an ordinate drawn from that point of intersection in the transformed curves, corresponding to the intersection of the given curves.

[Prop. 25.] Let  $KG, KH$ , be (Fig. 14) two tangents meeting in  $K$ ,  $yx$  the third tangent meeting  $ba$  in  $y$ . Let the lines  $KH, KG$ , be projected in  $ki, lh$ ; and  $yb, yx$ , into  $kl, ih$ ; then you have the parallelogram  $hikl$ ; then proceed according to the proposition.

[Ib. For by the properties of the conic sections] (by cor. 4, prop. 44, Conics 1).

[Ib. But according as the points.] (Fig. 15) This is plain from the nature of the ellipsis and (Fig. 16) hyperbola; and the figure cannot be a parabola, by reason of two parallel tangents  $ih$  and  $kl$ .

[Lem. 23.] The lemma is universal, as will appear by applying the demonstration to this figure 17.

[Lem. 24, from the nature of the conic sections] (by prop. 46, b. I. of my Conic Sections).

[Ib. cor. 1.] This holds as well (Fig. 18) when the tangent FG is on the other side.

[Lem. 25, also KH is to HL] (by cor. 1, lem. 24) for the tangents FH, LH, cut the parallel tangents ML, IK, in F and K.

[Lem. 25, cor. 2.] This (Fig. 19) holds as well when qe is on the other side of the figure; for in all cases (by cor. 1)  $KQ \times ME$  is given wherever the points Q, E fall, as suppose in q, e; for (by cor. 1, lem. 24)  $Bq : AM$ , or  $BK : : eI : eM$ ; and, by division,  $Kq : BK : : MI : eM$ . And  $Kq \times eM = BK \times MI =$  (by the prop.)  $KQ \times ME$ ; and  $KQ : Me :: (Kq : ME ::) Qq : Ee$ .

[Ib. cor. 3.] for since  $eM : ME :: QK : Kq$ ; therefore (by lemma 23) if the right lines  $eQ$ ,  $MK$ , and  $Eq$ , be drawn, the points of bisection will be placed in a right line given in position.

[Prop. 27, sch. describe the circle  $BKGC$ .] for the angle  $BKC$  (which is equal to the sum of the given angles  $PBK$  and  $KCP$ ) is always given.

[Ib. and when those other legs  $CK$ ,  $BK$ ] For it must be observed, that when the lines  $BP$ ,  $CP$ , touch the curve at an infinite distance, that these lines are parallel to one another, and to the asymptote. Then, to know the position of the asymptote, as the lines  $Bk$ ,  $Ck$  revolve round the circle, the intersection  $k$  will sometimes fall into the line  $MN$ , as at the point  $N$ ; then  $BN$  is parallel to one asymptote; for the intersection must necessarily fall in the line  $MN$ . And for the same reason the line  $BM$  will be parallel to the other asymptote. And, therefore, if the angle between them be bisected by the line  $OH$  (which is done by the pérp.  $OH$ ), then that line is the greater axis, or parallel to it.

[Ib. sch. But the squares of the axes] For the angle  $NBM =$  angle between the asymptotes =  $NLM$ , and  $NLH =$  half the angle of the asymptotes. Therefore  $LH$  is to  $HN$  as the transverse to the conjugate. And the squares of these axes are as  $LH^2$  to  $HN^2$ , or as  $HN^2$  to  $HK^2$ , that is, as  $LH$  to  $HK$ .

[Ib. There are also other lemmas] For if the sections (Fig. 20) are similar, and in similar position, and concentric, the

tangent acb in c is parallel to the tangent dxe in x, and therefore ab the ordinate is bisected in c the point of contact.

[Lem. 26, cor.] This is the same with prob. 32, of the Universal Arithmetic.

[Lem. 27, cor. in the construction] for then  $IH : HF :: (iX : XY ::) ih : hf$ . And  $IH : HG :: (iL : LM ::) ih : hg$ . Or, if it be made as  $iL :: LM :: IH : HG$ , it will be  $ih : hf :: IH : HF$ ; therefore, on the contrary, if it be  $ih : hf :: (iE : EV :: iX : XY ::) IH : HF$ , it will be  $iL : LM :: IH : HG$ , which comes to the former construction. For the solution of prop. 22, 23, 24, 25, 26, 27, see prop. 70, 71, 72, 73, 74, 75; b. III. my Conic Sections.

[Prop. 29, sch.] Make also KA to AS (Fig. 21), and LT to AT, as HG to GF, and draw MS, NA. Then the figures SAKM and ATLN are similar to FGHI; and since three of the angles S, A, K, or A, T, L, are in the proper lines CB, ED, DB, if the fourth angle M or N was in the fourth line EC, the problem would be rightly constructed. Therefore it is plain its place can be no where but where the line MN intersects EC, as at i, which is the place of the angle I.

Now we are to prove that PQ cuts BA in f, where F is to be placed. The triangle FGI is similar to PEi (by construction), and suppose them similar to fgi; then the triangles Pfi and Egi are also similar; for the angles at i are equal, and the sides about these angles proportional; therefore the angle Egi = Pfi; and since goi = Qof, oQf or PQF will be = fig. So that to have fig similar to FIG, PQF must intersect Eg in Q, to make the  $\angle PQE = \angle FIG$ ; and the rest follows of course.

## SECTION VI.

[Prop. 30, cor. 1.] For the times are as the areas, that is, as  $\frac{4}{3}GH \times AS$  to  $\frac{2}{3} \times AS \times 2AS$ .

[Ib. cor. 2.] For wherever the point P falls, viz. infinitely near A, still  $\frac{4}{3}GH \times AS$  (= area APS) =  $\frac{1}{2}AP \times AS$ . And  $8GH = 3AP$ . Therefore GH or the velocity of H : AP, or the velocity of P :: 3 : 8. But the velocity of H is every where the same, for always  $\frac{4}{3}GH \times AS = \text{area ASP}$ . And GH  $\propto$  ASP  $\propto$  time of describing AP.

[Ib. cor. 3.] For AP is the chord of a circle passing through A, S, P, and whose centre is H.

[Prop. 31, as GK the difference] For when F comes to touch the line GH, the point A will be distant towards G from the line FQO (which will then be  $\perp$  to GH) by the sine of AOR or AOQ, and is then at L; therefore  $GK = \text{arc } GF - \text{sine of the arc } AQ$ . And GK is as the time, and so it was in the construction; therefore the point P is rightly found.

[Ib. schol. but since] This is demonstrated in Keil's Astron. Lectures, p. 289, 297 (Fig. 22), or thus: Let AON = N (Fig. 23).

Since  $(AB : SH ::) OQ : OS :: 57,29578 : B$ , therefore  $B = OS$  in degrees of the circle AQ. And since  $(\text{rad.} : \text{sine of } AOQ ::) SO : SF :: B : D$ , then will  $SF = D$  in degrees. Let q be the true place of the body; Q the assumed place. Now since the time is as the area SAQ = OAQ

$$+ SOQ = \overline{QA + SF} \times \frac{OA}{2}; \text{ therefore the time is as}$$

$AQ \pm SF$ ; and, therefore, nearly as  $AQ \pm D$ , but accurately as  $Aq \pm SE$ . Take  $N\phi = D$ . Now  $OE : OQ :: (LE \text{ or } SE - SF \text{ or } Nq \pm N\phi)$ , or  $q\phi$  or  $Q\phi - qQ : Qq$ . And  $Q\phi : Qq :: QE : OQ$ , by composition, and because  $QE = OQ \pm OE$ .

But, by construction,  $OQ = \frac{OS \times L}{R}$ . And rad.

$(R) : \text{cos. of } AOQ :: SO : OF \text{ or } OE$ , therefore  $OE = \frac{SO \times \text{cos. } AOQ}{R}$ . Wherefore  $Q\phi : Qq :: (QE =) \frac{OS \times L}{R}$

$\pm \frac{SO \times \text{cos. } AQ}{R} : \frac{OS \times L}{R} :: L \pm \text{cos. } AQ : L$ . But  $N - AOQ + D = (AN - AQ + N\phi =) Q\phi$ . Therefore  $Qq = E$ , and  $AOq = AOQ + QOq = AOQ + E$ , nearly; and therefore E is rightly found. And, repeating the same work with these new angles, there will be found the angles F, G, H, I.

[Ib. the area AIKP will be given] (Fig. 23) by prop. 87, cor 2, and schol. my Conic Sections, b. II. Also  $AIKP = OPA$ , by cor. 1, prop. 86, ib.

Also area described = ASQ = ASP + PSQ = ASP + PQ  $\times \frac{1}{2}SN$  (nearly) = A (by supposition); therefore A — ASP =  $\frac{1}{2}SN \times PQ$ ; and  $PQ = \frac{2A - 2ASP}{SN}$ , nearly.

[Ib. and by such computations.—But the particular (Fig. 24) calculus] with the radius 1 and centre H describe the circle RSE; and draw SP, Sp, HPT, Hpf. Let fall the perpendiculars pt, pu, upon the lines HP, SP; which will be equal, because the angles pPt (HPB) and pPn are equal.

Let the arc RT = z, sine TQ = s, cos. HQ = x, SP = y, HP = v, AO or OB = a, SO or OH = n, OD = c, T = mean anomaly, l = *latus rectum*.

By similar sectors, HT (t) : HP (v) :: Tf (z) : pt or pn = vz. And the area  $\frac{SP \times pu}{2} = \frac{vyz}{2}$  = fluxion of BPS. But

(cor. 2, prop. 23, ellipsis)  $vy = cc + \frac{nn}{cc} \times PM^2$ , and (cor.

3, prop. 72, ib.)  $HP$  or  $v = \frac{cc}{a + nx}$ , and rad. (1) : PH

$\left(\frac{cc}{a + nx}\right) :: s : PM = \frac{cc s}{a + nx}$ , therefore,  $vy = cc +$

$\frac{nncss}{a + nx^2}$ , and area  $SPP = \frac{1}{2}ccz + \frac{\frac{1}{2}nncssz}{aa + 2anx, \&c.} = \frac{1}{2}ccz +$

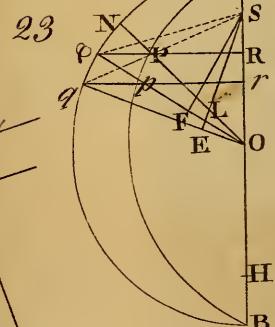
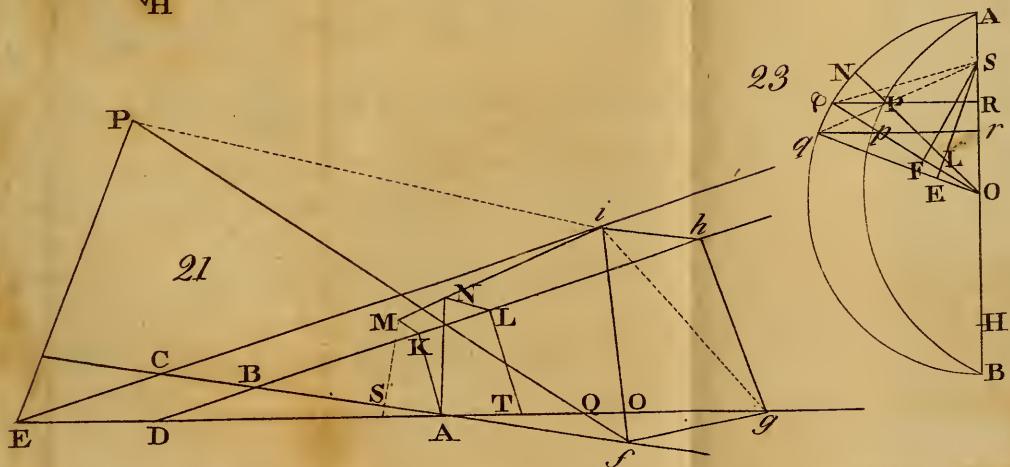
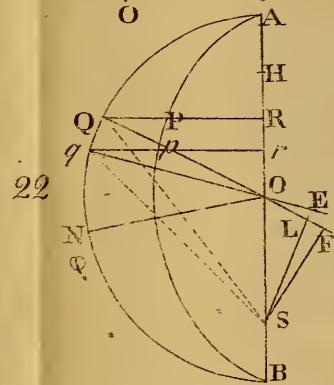
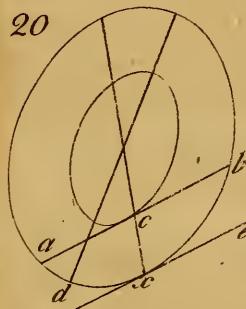
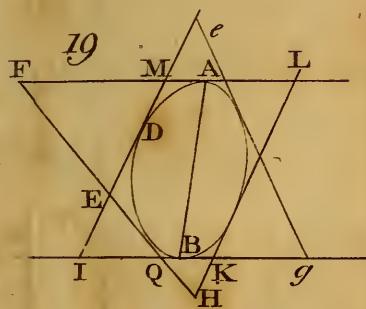
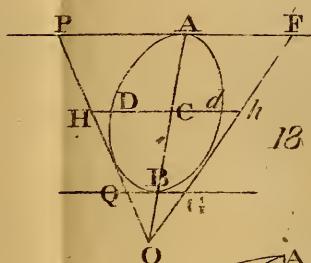
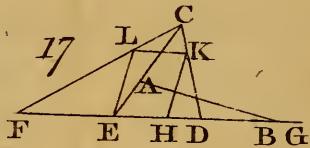
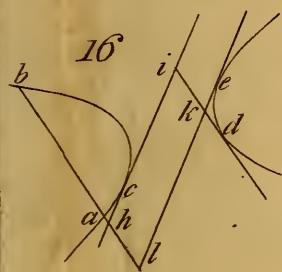
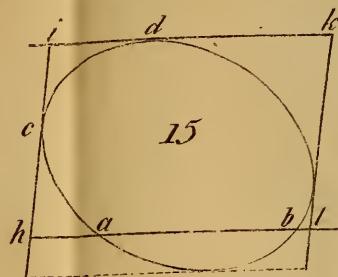
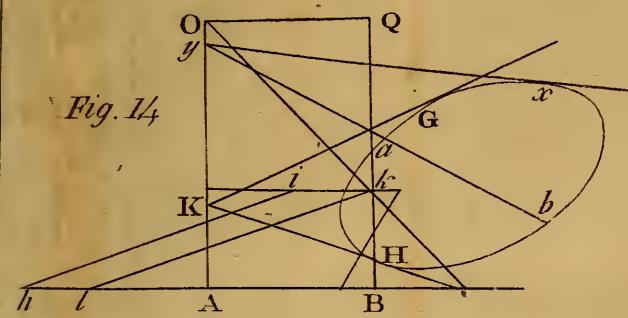
$\frac{nnccssz}{2aa} - \frac{n^3c^2s^2xz}{a^3}$ . But  $sz = -\dot{x}$ , and  $xz = \dot{s}$ , and  $\frac{cc}{a}$

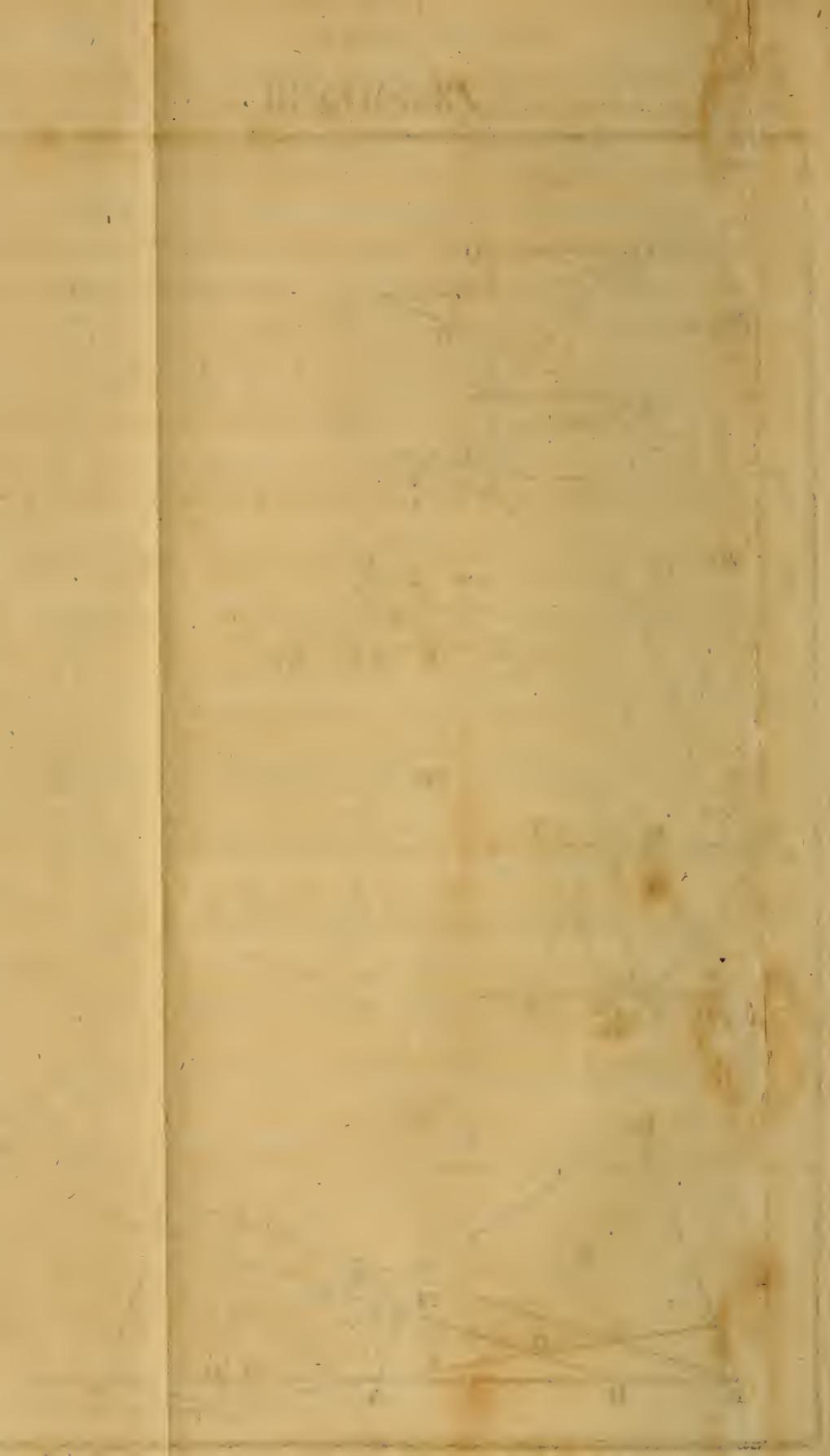
$= \frac{1}{2}l$ ; therefore the area  $SPP = \frac{1}{4}alz - \frac{\frac{1}{4}nnlsx}{a} - \frac{n^3ls^3}{2aa}$ ,

and the fluent  $BSP = \frac{alz}{4} - \frac{nsl}{4a} \times$  area EHQT —  $\frac{n^3ls^3}{6aa}$ ;

and, corrected,  $BSP = \frac{alz}{4} + \frac{nsl}{4a} \times$  area RTQ —  $\frac{n^3ls^3}{6aa}$ .

But  $RTQ = \frac{1}{2}z - \frac{1}{2}xs$ . Whence  $BSP = \frac{1}{4}alz + \frac{\frac{1}{4}nnl}{a} \times$   
 $\frac{\frac{1}{2}z - \frac{1}{2}xs}{\frac{1}{2}z - \frac{1}{2}xs} - \frac{n^3ls^3}{6aa} = \frac{2aa + nn}{8a} lz - \frac{nnls}{aa} \times \frac{ax}{8} + \frac{nss}{6}$ .





And dividing,  $z - \frac{8nns}{2a^3 + ann} \times \frac{\overline{ax}^{17}}{8} + \frac{nss}{6} = T$ , the mean anomaly. Whence  $z = T + \frac{nn}{4aa + 2nn} \times S.2T + \frac{4n^3}{6a^3 + 3ann} \times \overline{S.T}^3$ , because  $T$  is nearly  $= z$ , and  $2xs = S.2T$ . But since  $n$  is very small by supposition,  $z = T + \frac{nn}{4aa} \times S.2T + \frac{2n^3}{3a^3} \times \overline{S.T}^3$ . Where the quantities  $\frac{nn}{4aa} \times S.2T$ , and  $\frac{2n^3}{3a^3} \times \overline{S.T}^3$  are small arcs to be added to  $T$ .

Now  $D = c - \frac{cc}{a} = \frac{c}{a} \times \overline{a - c}$ ; and  $D \times \overline{AO + OD} = \frac{c}{a} \times \overline{aa - cc} = \frac{cnn}{a} = nn$  nearly, because  $c = a$ , very near. Therefore  $s.Y : \text{rad. (1)} :: nn : 4aa$ , and  $s.Y$  or  $Y$  (in small arcs)  $= \frac{nn}{4aa}$ ; also  $V : Y :: ST : \text{rad. (1)}$ , and  $V = Y \times S.2T = \frac{nn}{4aa} \times S.2T$ , which is our first term, or his first equation.

Again;  $S.Z : \text{rad. (1)} :: 4nD$ , or  $\frac{4nc}{a} \times \overline{a - c} : 3AO^2$ , or  $3aa$ , and  $S.Z = \frac{4nc}{3a^3} \times \overline{a - c}$ . But  $\overline{a + c} \times \overline{a - c} = aa - cc = nn$ , and  $a - c = \frac{nn}{a + c} = \frac{nn}{2a}$  nearly; therefore  $S.Z = \frac{4nc}{3a^3} \times \frac{nn}{2a} = \frac{2n^3c}{3a^4} = \frac{2n^3}{3a^3}$ , nearly. Also  $X : Z :: \overline{S.T}^3 : \text{rad. (1)}$ . And  $X = Z \times \overline{S.T}^3 = \frac{2n^3}{3a^3} \times \overline{S.T}^3$ , which is the second term, or his second equation. And when  $T$  is above  $90^\circ$ , then  $S.2T$  and  $V$  will be negative, and  $\angle BHP = T + X \pm V$ .

He calls  $Y$  or  $\frac{nn}{4aa}$  the greatest first equation, because it is greatest when  $S.2T = 1$ , and  $T = 45^\circ$ ; and  $Z$  or  $\frac{2n^3}{3a^3}$  the

greatest second equation, because the greatest it can be is when  $\overline{ST}^3 = 1$ , or  $T = 90^\circ$ .

### SECTION VII.

[Prop. 33, to AB the principal semi-diameter] this should be  $\frac{1}{2}AB$ .

[Ib. cor. 2] for the two first terms of the prop. are in a ratio of equality, and so the two last.

[Prop. 34. For (by cor. 7, prop. 16)], and cor. 6, prop. 4.

[Prop. 35. The same things supposed] to wit, that the space CS is as the area SDE of the circle, rectangled hyperbola, or parabola.

[Prop. 37, as appears by prop. 34] and cor. 7, prop. 16.

[Prop. 38, acquire the velocity CD] (Fig. 25) by prop. 10, cor. 2. The periodical times of the ellipses AD and AP are equal. The time of describing AP is (as APS, that is, as ADS or) as AD. Let the ellipsis AP coincide with AC, and the time of describing AC will still be as AD. Farther; draw cd  $\parallel$  to CD; because the time of describing the whole ellipses ADD, APP, are equal, therefore in equal times they describe areas which are as the whole ellipses, that is, as CD to CP, or ADS to APS. Wherefore in the time D describes AD or Dd, P describes AP or Pp, and (when P coincides with C) C describes AC or Cc; therefore the velocity of D : to velocity of C :: is as Dd : Cc :: or as SD to CD; but SD and the velocity of D is given; therefore the velocity of C is as CD.

[Prop. 39, cor. 2, 3.] In these cor. the line PD is the space the body would ascend to, or fall from (to acquire the velocity it is projected with), by a uniform centripetal force, according to cor. 1.

### SECTION VIII.

[Prop. 40, cor. 1.] This is evident, by supposing ITK convex towards C.

[Ib. cor. 2.] For by fluxions, let  $AC = P$ ,  $CD = A = x$  (Fig. 26). Then  $DG \propto x^n - ^1$  (by hyp. and prop. 39), and  $DEFG =$  fluxion of the area  $\propto x^n - ^1\dot{x}$ . And the area

$DChG \propto \frac{x^n}{n}$ . And when  $x$  becomes  $= p$ , the area  $AChB$   $\propto \frac{p^n}{n}$ . Therefore  $ADGB \propto \frac{p^n - x^n}{n}$ . And, therefore, by prop. 39, the velocity in D, at the distance A or  $x$ , is  $\propto \sqrt{\frac{p^n - x^n}{n}}$ , or (because  $n$  is given, and  $x = A$ ) as  $\sqrt{P^n - A^n}$ .

N. B. If the force is reciprocally as the distance, the curve  $BGh$  will be a rectangled hyperbola to the asymptotes  $AC$ ,  $Ch$ ; for  $DG \propto \frac{1}{CD} =$  (suppose to)  $\frac{bb}{CD}$ . And  $DG \times CD = bb$ , which is the known property of the hyperbola. And the velocity at C will be infinite, for the area  $ABhC$  is infinite. And the velocity at any place D is as the hyperbolic area  $ABGD$ , which may be found by Stone's Fluxions, p. 54; or by cor. 2, prop. 87, book II. my Conic Sections.

[Prop. 41. — in the trajectories found.] One being given to find the other.

[Ib. given the circle VR] that is, CV is known.

[Ib. in the least given time,] these small parts of time are taken constant and invariable.

[Ib. and the triangle ICK] that is, the triangle will be invariable.

[Ib. and suppose the magnitude of Q,] This quantity is any constant quantity, but unknown. Suppose in some case  $\sqrt{ABFD} = b \times IK$ , and  $Z = b \times KN$ . And in all cases (by prop. 40)  $\sqrt{ABFD} = b \times IK$ ; and in all cases  $Z$  or  $\frac{Q}{A} = b \times KN$ , that is, in all cases  $Q = b \times A \times KN$ ; which is plain, because Q and  $A \times KN$  are constant quantities: therefore if it be once  $\sqrt{ABFD} : Z :: KI : KN$ , or  $b \times IK : b \times KN :: KI : KN$ , it will always be so.

In this prop. the line CE is indetermined; and if Q were known, the areas of the curves abz and acx might be known to any distance CE.

[Ib. cor. 3.] In Fig. 4, Newt. Let CA = a, CV = r, CD = x, DF = y,  $\sqrt{aa - rr} = n$ .

Then the force DF being as  $\frac{1}{x^3}$ , the fluxion of the area ABFD is  $\frac{-x}{x^3}$ , and the fluent as  $\frac{1}{2x^2}$ ; and corrected, the fluent is  $= \frac{aa - xx}{2axxx} =$  area ABFD. Therefore the velocity at D is as  $\frac{\sqrt{aa - xx}}{ax}$ . Then,

1. Suppose a infinite, and  $x = r$ , then the area  $\sqrt{ABFD}$  becomes  $\frac{\sqrt{aa}}{ar} = \frac{1}{r}$ . And since at V the orbit IV is perp. to CV, therefore IK = KN, and  $\sqrt{ABFD} = Z = \frac{Q}{x}$ , that is,  $\frac{1}{r} = \frac{Q}{r}$ , and  $Q = 1$ , and  $Z = \frac{1}{x}$ ; also  $\sqrt{ABFD} = \frac{\sqrt{aa - xx}}{ax} = \frac{a}{ax} = \frac{1}{x}$  at any place I; and since  $\sqrt{ABFD} = \left(\frac{1}{x}\right) : Z \left(\frac{1}{x}\right) :: IK : KN$ , therefore TK = KN, and IN = 0; therefore in this case the orbit is a circle, as VXR.

2. If a be less than infinite, and since at V the orbit is perp. to CV as before, therefore IK = KN, and  $\sqrt{ABFD} = \frac{Q}{r}$ , that is,  $\frac{\sqrt{aa - rr}}{ar} = \frac{Q}{r}$ , or  $\frac{n}{ar} = \frac{Q}{r}$ , and  $Q = \frac{n}{a}$ . Whence  $Z = \frac{n}{ax}$ . Therefore  $D_b = \frac{n}{a} \times \frac{1}{\sqrt{\frac{aa - xx}{2axx} - \frac{nn}{aaxx}}} = \frac{n}{2a}$   
 $\sqrt{\frac{aaxx}{rr - xx}} = \frac{nx}{2\sqrt{rr - xx}}$ . And  $D_c = \frac{rr}{nx} \times D_b = \frac{nr}{2x\sqrt{rr - xx}}$ , and flux. of the area VacD =  $\frac{nrrx}{2x\sqrt{rr - xx}}$   
 $=$  fluxion of the sector VCX, and the Fl. :  $\frac{nrrx}{2x\sqrt{rr - xx}}$   
 $=$  sector VCX, which is as the angle VCX.

But, putting semi-conjugate = c, CH = Z, the hyperbolic sector VCR = Fl. :  $\frac{rcz}{2\sqrt{zz - rr}}$ , and CT =  $\frac{rr}{z} = x$ ,

and  $z = \frac{rr}{x}$ . Then instead of z and  $\dot{z}$ , putting their values, we shall have the hyp. sector VCR = Fl. :  $\frac{-rrcx}{2x\sqrt{rr - xx}} = \frac{c}{R}$   $\times$  cir. sector VCX. Therefore when  $x = CT$ , the hyp. sector VCR : cir. sector CVX :: c : n; that is, because the angle VCX is as the sector CVX, the angle VCX is to the hyper. sector VCR in a given ratio. And since CT or x continually decreases as the sector VCR increases, the body P draws continually nearer the centre C.

3. If the velocity be greater than falling from an infinite height, the flux. area ABFD =  $\frac{+x}{x^3}$ , and the fluent  $\frac{-1}{2xx}$ .

But at first the area (suppose) =  $\frac{bb}{2}$ , and  $x = a$ , an infinite line; therefore  $ABFD - \frac{bb}{2} = \frac{1}{2aa} - \frac{1}{2xx}$ , and

$ABFD = \frac{bb}{2} + \frac{xx - aa}{2aaxx}$ . Therefore  $\sqrt{ABFD}$  is as  $\sqrt{\frac{bbaaxx + xx - aa}{aaxx}} = (\text{by substitution}) \sqrt{\frac{sxx - aa}{aaxx}}$ .

But in V,  $x = r$ , and  $\sqrt{ABFD} = \sqrt{\frac{srr - aa}{aarr}} = \frac{Q}{r}$ , and

$Q = \sqrt{\frac{srr - aa}{a}}$ , and  $Z = \frac{Q}{x} = \sqrt{\frac{srr - aa}{ax}}$ . Then  $D_b$

$$= \frac{\sqrt{srr - aa}}{2a\sqrt{\frac{sxx - aa}{aaxx}} - \frac{srr - aa}{aaxx}} = \sqrt{\frac{srr - aa}{s}} \times \frac{x}{2\sqrt{xx - rr}}$$

$$= (\text{by substitution}) \frac{Ax}{2\sqrt{xx - rr}}, \text{ and } D_c = \frac{rr}{xx} \times D_b =$$

$$\frac{Arr}{2x\sqrt{xx - rr}} ; \text{ therefore, } \frac{Arrx}{2x\sqrt{xx - rr}} = \text{flux. area VacD},$$

or of VCX. And the Fl. :  $\frac{\text{Arrx}}{2x\sqrt{xx - rr}}$  = factor VCX,  
which is as the angle VCX.

But in the ellipsis VRS let CH = z (Fig. 27), semi-conjugate = c. Then the elliptic sector VCR = Fl. :  $\frac{rcz}{2\sqrt{rr - zz}}$ , and CT =  $\frac{rr}{z} = x$ , and  $z = \frac{rr}{x}$ , and  $\dot{z} = \frac{-rrx}{xx}$ . Then putting for  $z$  and  $\dot{z}$  their values, and the ellip.

sector VCR = Fl. :  $\frac{-crrx}{2x\sqrt{xx - rr}}$  =  $\frac{c}{A} \times \text{cir. sector VCX}$ ;

therefore when  $x = CT$ , the ellip. sector VCR : cir. sector VCX :: c : A ; that is, because the angle VCX is as the sector VCX, the angle VCX is to the elliptic sector VCR in a given ratio. And since CT or x continually increases as the sector VCR increases, the body P goes continually farther and farther from the centre C.

[Ib. and the centripetal force becoming centrifugal] for then the curve VPQ will turn upwards; and, the law of the force being the same, it will be constructed the same way by the elliptic sectors, taking the point A between C and V. And here the velocity will increase as it recedes from the centre C. But it can never revolve round this centre.

### SECTION IX.

[Pr. 44, or in *antecedentia* with a celerity] for then the line mnC falls beyond s, sr being = rk, and sCk = 2rCk; and consequently the point m falls without the circle.

[Ib. and with a less force] if the orbit moves slower in *antecedentia* than with twice the celerity of CP in *consequentia*; for then the point m falls within the circle, between r and s.

The meaning of this prop. is this,---that the difference of the forces at different distances from the centre are reciprocally as the cubes of the distances (which forces are requisite to make the body move in a quiescent or revolving orbit, and the distances to be the same in both).

[Ib. cor. 1.] For mn represents the difference of the forces by which p revolves to n, or P to K, in equal times; and the versed sine of RK represents the force whereby a body moves from R to K in the circle in the same time.

[Ib. cor. 2. as half the *latus rectum*] for these forces are as bn or ar to bp or as. But  $ar = \frac{nr^2}{rD} = \frac{ba^2}{aD}$ . And  $asD : sp^2 : : aD : 2R$ ; and  $asD = \frac{sp^2 \times aD}{2R} = \frac{ab^2 \times aD}{2R}$ ; and  $as = \frac{ab^2 \times aD}{sD \times 2R} = \frac{ab^2}{2R}$ . Whence these forces are as  $\frac{ba^2}{aD}$  to  $\frac{ba^2}{2R}$ , or as R to aC or CV, in Fig. 2, Pl. 18.

[Ib. cor. 3.] For let the force in the immoveable ellipsis be  $\frac{FFA}{T^3}$ , and in V it will be  $\frac{FFT}{T^3}$  or  $\frac{FF}{TT}$ . And the force by which a body may revolve in a circle at the distance CV is  $\frac{FFR}{T^3}$ . And the difference of the forces in V (of revolving in the moveable and immoveable ellipsis) is  $\frac{GGR - FFR}{T^3}$ ,

and in every altitude A is  $\frac{GGR - FFR}{A^3}$ ; and the force in the moveable ellipsis will be  $\frac{FFA}{T^3} + \frac{RGG - RFF}{A^3}$ .

[Ib. cor. 4.] After the same manner as in the two foregoing corollaries; because when the velocity is given, the force  $\propto$  curvature or reciprocally  $\propto$  radius of curvature; therefore  $T : R : : \frac{VFF}{TT} : \frac{RVFF}{T^3}$ , which is the force whereby a body may revolve in a circle at the distance CV. And, by cor. 1,  $FF : GG - FF : : \frac{RVFF}{T^3} : \frac{RVGG - RVFF}{T^3}$ , which is the difference of the forces (in the moveable and immoveable orbits) in the vertex V. And by the prop.  $\frac{1}{T^3} : \frac{1}{A^2}$

$:: \frac{RVGG - RVFF}{T^3} : \frac{RVGG - RVFF}{A^2}$ , which is the dif-

ference of forces in every altitude A ; and therefore the force in the moveable trajectory upk will be  $X + \frac{RVGG - RVFF}{A^3}$ .

[Ib. cor. 6.] In this cor.  $X = 0$ , and  $RVGG - RVFF$  is compounded of all given quantities, and the force in Vpk will then be  $\propto \frac{1}{A^3}$ . This also appears from the prop. for the difference of the forces in P and p is as  $\frac{1}{Cp^3}$ , but the force in one of them, viz. at P, in the line VP, is 0 ; and therefore the other force in Vpk is the difference of the forces, and is as  $\frac{1}{Cp^3}$ . Also this curve is the same with (Fig. 28) that in cor. 3, pr. 41 ; for let vad be a circle ; and by this cor. 6,  $\angle vcp \propto \angle vcz \propto \text{arc } va \propto \text{area } vca \propto \text{area } vcb$ , which is the construction in cor. 3, pr. 41 ; also  $cp = cz$  or  $cx$ , because  $vz$ ,  $ax$ , and  $bx$  are tangents, which is also the same construction as in cor. 3, pr. 41.

[Pr. 45 (Fig. 29), but orbits acquire the same figure] Let the small parts of the curve df, fh, hp, &c. be described by a body A in indefinitely small given parts of time. And let another body B go from d in the same direction de, and with a velocity which is to the velocity of the former in the sub-duplicate ratio of the centripetal force of B to that of A ; and let it arrive at s in the aforesaid small given part of time. Now since  $ef : rs :: \text{force of A} : \text{force of B} :: \text{velocity}^2 \text{ of A} : \text{vel.}^2 \text{ of B} :: de^2 : dr^2 :: \text{square of the time of B's moving through } de : \text{square of the time of B's moving through } dr$ ; therefore, by lem. 10, the body B will pass through f. And since  $rs$  is every where as  $rd^2$ , it is evident the same curve dsf will be described by both bodies, and which  $de$ ,  $fg$  touches in  $d$ ,  $f$ . Also the velocity of A in f : vel. of A in d :: as perpendicular from C on de : to  $\perp$  on fg :: vel. of B in f : vel. of B in d :: And vel. of A in f : velocity of B in f :: vel. A in d : vel. of B in d ::  $de : dr$ ; that is, in a given ratio ; that is,  $fg$  : to  $fk$ , and  $hn$  to  $hl$ , and  $px$  to  $pt$  are in a given ratio ; and therefore when A is arrived at g, B would be at k; and since  $gh : ki :: gf^2 : kf^2 :: ed^2 : rd^2 :: ef : rs$ ; or  $gh : ef :: ki : rs$ ; that is, the centripetal of A to that of B at equal

distances being in a given ratio, B will pass through i, and consequently through h. After the same manner it may be proved that the body B will pass through the points m, p, u, y, and describe the same curve with the body A.

Farther; if another body D move through  $\beta\gamma \parallel$  to de, and with a velocity which is to the velocity of A as  $C\beta$  to  $Cd$ , and be acted upon by a force which in the points  $\beta, \delta, z, \&c.$  is to the force in the points d, f, h, &c. respectively, as  $C\beta$  to  $Cd, C\delta$  to  $Cf, Cz$  to  $Ch, \&c.$ ; then, I say, the curve described by the body D, viz.  $\beta\delta z\varphi\omega$ , will be similar to the curve dfhpy; for in the time that A would arrive at e, D would arrive at  $\gamma$ ; and because  $C\gamma : Ce :: C\beta : Cd ::$  force at  $\beta$  : force at d  $:: \gamma\delta : ef :: C\gamma - \gamma\delta$  or  $C\delta : Ce - ef$  or Cf; and  $C\beta : C\delta :: Cd : Cf ; \gamma\delta$  is every where as  $\gamma\beta^2$  in all the points between  $\beta$  and  $\gamma$  (by lem. 10), it is manifest the fig.  $C\beta\delta$  is similar to the fig. Cdf, and the tangent  $\delta\varepsilon \parallel$  to fg; and because the areas  $Cgf : Cfd :: C\epsilon\gamma : C\delta\beta$ ; wherefore when A comes to g, D comes to  $\epsilon$ ; but  $gh : \epsilon z :: Cf : C\delta :: Cg : C\epsilon :: Ch : Cz$ , as before; and  $Cf : Ch :: C\delta : Cz$ , whence the fig.  $C\delta z$  is similar to  $Cfh$ ; and after the same manner it will be proved that the nascent figures  $Cz\varphi, C\varphi\omega$ , are similar to Chp, Cpy; and therefore the whole figure  $\beta\delta z\varphi\omega$  is similar to the figure dfhpy.

Or universally, if the orbits  $\beta\delta\varphi$  and  $d\delta p$  are similar,  $C\beta \propto \gamma\delta \propto$  force  $\times$  time<sup>2</sup> of describing  $\beta\delta \propto$  force  $\times \frac{\beta\delta^2}{\text{velocity}^2}$   
 $\propto$  force  $\times$  (Fig. 30)  $\frac{C\beta^2}{\text{velocity}^2}$ . Therefore velocity<sup>2</sup>  $\propto$  force  $\times$  distance  $C\beta$ . Whence if  $C\beta$  be given, velocity  $\propto \sqrt{\text{force}}$ ; and if velocity  $\propto C\beta$ , force  $\propto C\beta$  also, which agrees with what went before. If the force be given, velocity  $\propto \sqrt{C\beta}$ . If velocity be given, force  $\propto \frac{1}{C\beta}$ .

[Ib. exam. 1, l. 3.] here  $\frac{T^3 - 3TTX + 3TX^2 - X^3}{A^3}$  is twice repeated in the English, which is wrong.

[Ib. By this collation of the terms] for since the same figure will be described by making the centripetal force pro-

portional at equal distances (altering the velocity in the sub-duplicate ratio of the force); and since, when  $R$  is nearly =  $T$ , and  $x$  very small, the centripetal force in the revolving ellipsis will be as  $RGG - FFx$ , and in this orbit (because in both the denominator  $A^3$  is the same) is as  $T^3 - 3TTx$ , therefore these forces ought to be proportional at all distances, viz. when  $x = 0$ , and any other indeterminate distance  $A$  or  $T - x$ ; wherefore (in these two cases)  $RGG : T^3 :: RGG - FFx : T^3 - 3TTx ::$  (and by division)  $FFx : 3TTx :: FF : 3TT$ , which is the construction of the problem. After the same manner, in example 2, the centripetal force in the revolving ellipsis, and this new orbit, will be as  $RGG - FFx$ , and  $T^n - nT^{n-1}x$ , and to have the orbits similar, putting  $x = 0$ ,  $RGG : T^n ::$  (putting  $T - x$  for any indeterminate distance)  $RGG - FFx : T^n - nT^{n-1}x ::$  (by division)  $FFx : nT^{n-1}x :: FF : nT^{n-1}$ . After the same manner, in example 3, it will be  $RGG : bT^m + cT^n :: (RGG - FFx : bT^m + cT^n - mbT^{m-1}x - ncT^{n-1}x) :: FFx : mbT^{m-1}x + ncT^{n-1}x$ . Or  $GG : bT^{m-1} + cT^{n-1} :: FF : mbT^{m-1} + ncT^{n-1}$ . The quantity  $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$  (in cor. 2, pr. 44) is universally as the centripetal force, whether the apsides move backwards or forwards; for  $G : F :: VCp : VCP :: (p. 119) pCn : pCk$ . And the force in the revolving ellipsis is greater than in the immoveable one, when  $n$  (Fig. 2) is without the line  $ks$ ; and in that case ( $F$  is less than  $G$ , and)  $\frac{RGG - RFF}{A^3}$  will be affirmative. But the force is less when  $n$  falls between  $k$  and  $s$ , for then ( $F$  is greater than  $G$ , and)  $\frac{RGG - RFF}{A^3}$  is negative, as it ought to be.

## SECTION X.

[Pr. 50, it is evident from the construction] Since  $VP$  touches the curve in  $P$ , and  $PB$  is  $\perp$  thereto, therefore  $B$  is the point of contact of the circle  $AD$ , and the wheel or generating circle, and therefore  $BV = AO$ . Also since  $WT$  is  $\perp$  to  $TV$ , therefore  $V$  is the point of contact

of the generating circle (whose diameter is VW) and OS; whence  $VW = OR$ . The figures are similar, because their axes are as the radii of the spheres.

[Pr. 52, hence since in unequal —] suppose TR to be a smaller oscillation, to find the velocity in I; take st : SR :: TI : TR; then velocity at R in the arc TR : velocity at R in SR :: TR : SR; and velocity at I in TR : velocity at R in TR :: (prop. 51) velocity at t in SR : velocity at R in SR ::  $\sqrt{SR^2 - tR^2} : SR$ . And, *ex equo*, velocity at I in TR : velocity at R in SR :: TR  $\sqrt{SR^2 - tR^2} : SR^2 :: \frac{TR}{SR}$   
 $\sqrt{SR^2 - tR^2} : SR ::$  (because  $SR : TR :: tR : IR$  as)  
 $\sqrt{TR^2 - IR^2} : SR$ .

[lb. There are obtained from the times given] for in two unequal arcs two corresponding parts are described in equal times (by prop. 51); therefore the velocities in these points will be as the entire arcs; and therefore both the velocities and arcs will be known.

[lb. And if the absolute force of any globe] that is, the force at a given distance be called V, then the force at the distance CO is  $CO \times V$ . But in a given time HY is as the force; whence  $CO \times V$  is as HY, and therefore HY is described in a given time.

[lb. cor. 1. For this time] for then  $AR = AC$ , and V is given : therefore, this time : time of  $\frac{1}{2}$  oscillation ::  $\sqrt{\frac{AC}{AC}} : \sqrt{\frac{AR}{AC}} :: 1 : \sqrt{\frac{AR}{AC}}$ .

[lb. cor 2. But in that case] for (in Fig. 2, Pl. 19), by cor. 1, 2, prop. 49,  $AS : Bv :: PS : PV :: 2CE : CB$  (in this case as) :: 2 : 1 :: (and by division)  $AP : BV - PV$ ; whence  $AP = 2BV - 2PV$ ; but versed sine of  $\frac{1}{2}PB = \frac{BV - PV}{2}$ , for cosine =  $\frac{1}{2}PV$ ; and  $\frac{1}{2}VB - \frac{1}{2}VP =$  versed sine.

[lb. as M. Huygens] all this is demonstrated in Keil's Philosophy. See my large book of Mechanics, prop. 40, cor. 4.

[Pr. 56, let the projection] T is projected into P, and t into p.

[Ib. as also its position] The greater axis  $2Tt$  is perpendicular to PO.

[Ib. and since the area PO $p$ ] for let (Fig. 31)  $\frac{\text{area OPP}}{OP} = sp$ , which is given ( $sp$  being  $\perp$  to PO); and  $PB^2 : Pr^2 :: PB^2 - SP^2 : PS^2$ ; whence Ps is given, and thence SO, the angle PO $p$ , Op, and Pp, the point p, and angle OPP.

### SECTION XI.

[Pr. 57, and about each other] considering either body as at rest.

[Pr. 58, by the same forces there may be described, &c.] that is, by making the velocity to the former velocity as ( $\sqrt{sp}$  to  $\sqrt{CP}$ , that is, as)  $\sqrt{P + S}$  to  $\sqrt{S}$ , as appears by the prop.

[Pr. 60, in a ratio sesquiplicate] it should be subsesquipuplicate.

[Ib. of the other ellipsis,] As  $S\frac{1}{3} : \overline{S + P}\frac{1}{3} :$  Let  $S\frac{1}{3} : a : e : \overline{S + P}\frac{1}{3}$  be  $\therefore$ . Then  $S\frac{1}{3} : \overline{S + P}\frac{1}{3} :: (S : a^3 : : )e^3 : \overline{S + P}$ . But (since  $\overline{S + P} : e^3 :: a^3 : s$ )  $e^3$  is the first of two mean proportionals between  $S + P$  and  $S$ .

[Pr. 61.] Note, if a body were placed in the centre of gravity, and whose force would be sufficient to cause one of the bodies to revolve around it, and to describe the same figure, yet it would not cause the other body (if it were unequal) to describe its figure, except in that law of centripetal force which is as the distance, where the periodic times are all equal. For, to preserve the same motion as before, either body must be attracted to the body in the centre of gravity with a force which is as the other body, or as its own distance, which is the case of one body attracting another, and holds only (in the case of their being attracted by a third body in the centre of gravity) in the law of centripetal force before-mentioned. And therefore in

[Pr. 62.] The foregoing note is to be observed.

[Prop. 64. This would be the case] For this only causes the bodies T, L, to revolve more swiftly round their centre of gravity D, but affects not the other bodies.

[Ib. with equal periodical times.] By the foregoing (in the three bodies S, T, L) the body S describes an ellipsis round C; and, by considering the centre of gravity of S, T, as describing an ellipsis (as before of T, L), it will appear the same way that the body L describes an ellipsis round C; and as the centre of gravity of S, L, describes an ellipsis, so T describes an ellipsis also round C. Farther; from what went before, the centripetal force of S towards C is as  $\overline{T + L} \times \overline{SC + CD}$ , or  $\overline{T + L + S} \times \overline{SC}$ . Also the centripetal force of T towards D is as  $L \times \overline{TD + DL} + S \times \overline{TD}$  (which latter part arises from the resolution of the force ST into SD, DT, and DT acts towards D), or as  $\overline{T + L + S} \times \overline{TD}$ . Now, by cor. 2, 8, prop. 4, the periodic times are in the subduplicate ratio of the radii directly, and the subduplicate ratio of the forces inversely; therefore the periodic time of S round C : periodic time of T round D ::

$$\checkmark \frac{SC}{SC \times \overline{T + L + S}} : \checkmark \frac{TD}{TD \times \overline{T + L + S}} ; \text{ and are}$$

therefore equal; so that after one revolution the bodies all return to their first places.

After the same manner the point C, and a fourth body V, as also the bodies S, T, L, will describe ellipses round their common centre of gravity B, for any one, and the centre of gravity of the other three will describe ellipses; and the case is the same if there were more bodies. Also, as before, and by a like resolution of forces, since the forces VS, VT, VL, are resolved into VC, CS; VC, CT; VC, CL; the former acting in the direction of CV towards V, the latter to C, their centre of gravity, and are as the distances of the bodies therefrom, as before, they will therefore still move round their centre of gravity C as before, but swifter, although all the four revolve round B. Also periodic time of V round B : pe-

riodic time of S round C ::  $\sqrt{\frac{BV}{S + T + L \times CB + BV}}$  :

$\checkmark \frac{SC}{T + L \times S \times SC + V \times SC}$  :: (that is, as)

$\checkmark \frac{SC}{T + L + S + V \times BV}$  ::  $\checkmark \frac{SC}{T + L + S + V \times SC}$  ;

and are therefore equal; and if more bodies were added, all the bodies would perform their revolutions in equal periodic times. Q.E.D.

[Prop. 66, cor. 2. Such is the force NM.] This force always acts from P in a direction parallel to TS, and from M towards T.

[Ib. cor. 6, the periodic time will be increased,] for then the periodic time is  $\propto \frac{\text{rad. } \frac{3}{2}}{\sqrt{\text{force}}}$ , which is a greater ratio than  $\text{rad. } \frac{3}{2}$ , because the force is diminished.

[Ib. cor. 7, the upper apsis to go backwards] for  $\frac{n^2}{m^2} - 3$

$> - 2$ . And  $\frac{n^2}{m^2} > 1$ ; therefore n is  $>$  than m. Also put D = distance (r = a given quantity), and the force is  $(\frac{D^3 + r^3}{D^2}$ , or rather) as  $\frac{1}{D^2 + \frac{1}{D}}$ , and when the distance increases, the decrease of the force is  $D^2 + \frac{1}{D}$ , which is less

than  $D^2 + \frac{D^3}{D}$ , which is the decrease according to the duplicate ratio. Also at the conjunction, &c. the force of P towards T is the centripetal force of T + LM — TM, or the centripetal force of T — KL. Also the decrease of the force is as  $d^2 - \frac{1}{d}$ , which is greater than  $d^2 - \frac{d^3}{d}$ , or the decrease according to the duplicate ratio. And  $\frac{n^2}{m^2} - 3 < - 2$ ; and n  $<$  m.

[Ib. cor. 7. The truth of this] for then, the farther P recedes from T, the more it is attracted towards the bodies S, S, and therefore less towards T than it would otherwise be.

[Ib. cor. 8, when the apsides are in the syzygies] for then NM is greater and LM less than before.

[Ib. cor. 9. Now therefore —] For the ratio of KL to LM is least when the apsides are in the quadratures, and greatest in the syzygies.

[Ib. For the forces LM] as is shewn in cor. 7.

[Ib. cor. 10, from the syzygies to the quadratures] for after the syzygies, the body P, by the action of S, is made to move in lines which successively cut the plane TPS at greater and greater angles. And for the same reason, at the  $90^\circ$  from the octants (between C and A), the body is drawn into lines (or little planes) which cut TPS at greater and greater angles, which before  $90^\circ$  cut it at less and less.

[Ib. and by a like reasoning] for the inclination is increased from u to D (Fig. 32), diminished from D to S, increased from S to C, diminished from C to u; as is plain by supposing T and any point p, wherein the body is, to be joined; and the plane pTt to revolve round pT, till it pass through the new place of the body p (which it acquires either above or below the plane pTt) by the (action of S upon it, or the) force NM; and then it will appear how that new acquired plane cuts the plane SCT, whether in a greater or lesser angle.

[Ib. cor. 11 — from the former plane CD] Since the inclination of the orbit is diminished from C to A, the intersection with TSE will move from C to B, and from D to A; and seeing that inclination is increased from A to D, the intersection (or nodes) will still move towards B and towards A.

[Ib. being always either retrograde] that is, considering a whole revolution; for in some points they go forward.

[Ib. cor. 13. And since the causes and proportions] to one another, and to the force of S, &c.

[Ib. cor. 14. But since the forces] For SK : LM :: accelerating force of T towards S : perturbating force of S. And therefore the perturbating force of S  $\propto$

LM × accelerating force of T towards S. • But since  $SK =$

$ST^2$  nearly, and  $LM$  (in its mean quantity)  $= PT$ , and the accelerating force of T towards S is as the body S directly, and  $ST^2$  reciprocally, therefore the perturbating forces  $LM, NM, \propto$   
 $\frac{PT \times \text{body } S}{ST^3}$ , that is,  $\propto \frac{PT \times \text{bodies}}{\square \text{ periodic time of T round S}}$ .

Or  $\propto PT \times \text{density of S} \times \text{cub. apparent diameter of S}$  (for the apparent diameter is as the real diameter directly, and the distance reciprocally).

[Ib. cor. 16, square of the periodical time of the body P conjunctly.] This holds as true in the synodic time of the body P (since the action of the forces that cause these errors begin and end at the quadratures, which comes to the same as if they begun and ended at the syzygies): see what follows.

[Ib. and hence the angular errors] these in one revolution are as the linear errors directly, and the radius or distance reciprocally; that is, as the forces and square of the time of revolution of P directly, and the distance TP reciprocally; that is (because the force  $\propto TP$ ), as square of the time of revolution.

[Ib. Let these ratios] The angular errors of P observed from T are as  $\frac{\text{forces} \times \text{time}^2}{TP}$ , that is (by cor. 14), as

$\frac{PT \times \text{body } S \times \text{time}^2}{PT \times \square T's \text{ periodic time}}$ , that is, in the time of one revolution of P (and if S be given), as  $\frac{\square \text{ time of P's revolution}}{\square \text{ periodic time of T}}$ .

[Ib. both these motions will be as the periodical time of the body P directly,] for the angular motion or velocity, or the mean angular errors, are as the sum of all the angular errors in any time directly, and the time reciprocally; that is (in the periodic time), as

$\frac{\text{body } S \times \square \text{ time of P's revolution}}{\text{time of P's revolution} \times \square \text{ periodic time of T}}$ , that is, as  $\frac{\text{body } S \times \text{time of P's revolution directly}}{\text{the square of the periodic time of T reciprocally}}$ .

[Cor. 20. And thence the greatest height of the water This is explained by Worster in his (Experimental) Philo-

sophy, page 72. Also see my Geography, prop. 12, sect. 1.

### SECTION XII.

[Prop. 79. As the lineola Dd,] when the radius of the sphere is given ; and when the radius is not given, as Dd and the radius conjunctly ; therefore AS, and also PS or PE, may be of any magnitude.

[Ib. the sum of all the rectangles  $PD \times Dd$ ] that is, as  $\frac{PF + PD}{2} \times DF$ , or  $\frac{PF + PD}{2} \times \overline{PF - PD}$ , or  $\frac{PF^2 - PD^2}{2}$ .

[Prop. 80. The whole force of the sphere will be as the whole area ANB.] Also it is manifest that the attraction or force of any part of the sphere FEB is as the correspondent part of the area DNB.

[Pr. 81, ex. 1.] The area SL into AB = SL  $\times$  AB, because both SL and AB are given. And the area LD into AB is a trapezoid, whose base is AB, and the two parallel fides LA and LB; for as the point D removes nearer A, the ordinate LD decreases (from B, where it is LB) to A (where it is

LA); and that area is  $\frac{LA + LB}{2} \times AB$ , or  $\frac{LA + LB}{2} \times \overline{LB - LA}$ , that is,  $\frac{LB^2 - LA^2}{2}$ ; or it is  $\frac{2LA + AB}{2} \times AB$ ,

that is, LS  $\times$  AB. And the ordinate  $\frac{ALB}{LD}$  is the ordinate of an hyperbola between the asymptotes whose centre is L, and one asymptote LB; for ALB is a given rectangle, and the nature of the hyperbola is that  $\frac{ALB}{LD}$  (LD the part of the asymptote being taken as the axis) is as the ordinate erected on D : whence the reason of the construction will be evident ; for aABb is the aforesaid trapezoid, by construction ; also in the hyperbola ab the rectangle LB  $\times$  Bb is given, but that (by construction) is = the rectangle ALB, when the hyperbolic area aABb is that described by the ordinate  $\frac{ALB}{LD}$ . And

the difference of that and the trapezoid (which is the difference of the two first areas) is the area aba required.

[Ib. Ex. 2, the third  $\frac{ALB \times SI}{2LD^2}$ ] this is easily calculated by fluxions. LD is a flowing quantity; the fluxion of the area is  $\frac{ALB \times SI}{2} \times LD - \frac{1}{2} \times LD\dot{D}$ ; and the whole area is  $\frac{ALB \times SI}{2} \times LD - \frac{1}{2}$ , that is,  $\frac{ALB \times SI}{2LD}$ , or (when LD = LB)  $\frac{ALB \times SI}{2LB}$ : from this take the area (when LD = LA, viz.)  $\frac{ALB \times SI}{2LA}$ , and the difference is the area sought,  $\frac{ALB \times SI}{2LA} - \frac{ALB \times SI}{2LB}$ .

[Ib. Ex. 3.] The fluxion of the first area is  $\frac{SI^2 \times SL}{\sqrt{2SI}}$   $\times LD - \frac{1}{2} \times LD\dot{D}$ , and the whole area is  $\frac{2SI^2 \times SL \times LD - \frac{1}{2}}{\sqrt{2SI}}$  or  $\frac{2SI^2 \times SL}{\sqrt{2SI}} \times \frac{\sqrt{LB}}{\sqrt{LA}}$ , and that part of the area on AB is  $\frac{2SI^2 \times SL}{\sqrt{2SI}} \times \frac{1}{\sqrt{LA}} - \frac{1}{\sqrt{LB}}$ . The fluxion of the second area is  $\frac{SI^2}{2\sqrt{2SI}} \times LD - \frac{1}{2} \times LD\dot{D}$ ; and the whole area  $\frac{SI^2}{\sqrt{2SI}}$   $\times LD\dot{D}$ ; and the area on AB is  $\frac{SI^2}{\sqrt{2SI}} \times \sqrt{LB} - \sqrt{LA}$ .

Also the fluxion of the third area is  $\frac{SI^2 \times ALB}{2\sqrt{2SI}} \times LD - \frac{1}{2}$   $\times LD\dot{D}$ ; and the whole area  $\frac{SI^2 \times ALB}{3\sqrt{2SI}} \times LD - \frac{1}{3}$ ; and the area on AB is  $\frac{SI^2 \times ALB}{3\sqrt{2SI}} \times \frac{1}{\sqrt{LA^3}} - \frac{1}{\sqrt{LB^3}}$ .

[Ib. And those after due reduction] LA, LI, LB, are  $\ddot{\dots}$ ; for SP =  $\frac{SA^2}{SI}$ , PI =  $\frac{SA^2}{SI} - SI$ , LI =  $\frac{SA^2 - SI^2}{2SI}$ . And

$LA = PS - AS - LI = \frac{SA^2 + SI^2}{2SI} - SA$ . And  $LB = \frac{SA^2 + SI^2}{2SI} + SA$ . But  $\frac{SA^2 + SI^2}{2SI} - SA : \frac{SA^2 + SI^2}{2SI}$   
 $\frac{SA^2 + SI^2}{2SI} + SA$ , are  $\ddot{\approx}$ , as will appear by multiplying them.

Hence  $\sqrt{LB} - \sqrt{LA} = \sqrt{2SI}$ . For  $2\sqrt{LB \times LA} = 2SI = 2LS - 2SI = LB + LA - 2SI$ . And  $LB + LA - 2\sqrt{LB \times LA} = 2SI$ . And (by evolution)  $\sqrt{LB} - \sqrt{LA} = \sqrt{2SI}$ . Therefore the first area ( $= \frac{2SI^2 \times SL}{\sqrt{2SI}} \times \frac{\sqrt{LB} - \sqrt{LA}}{\sqrt{LB \times LA}}$ )  $= \frac{2SI^2 \times SL}{LI}$ . And the second area  $= SI^2$ . And  $\sqrt{LB^3} - \sqrt{LA^3} (= LB + LA \times \sqrt{LB} - \sqrt{LA} + \sqrt{LB} \times \sqrt{ALB} - \sqrt{AL} \times \sqrt{ALB} = 2LS \times \sqrt{2SI} + \sqrt{ALB} \times \sqrt{2SI} = \frac{2LS + LI}{2} \times \sqrt{2SI}) = \frac{SL^2 \times ALB}{3\sqrt{2SI}}$   
 $\times \sqrt{2SI}$ . Whence the third area  $= \frac{SL^2 \times 3LI}{3\sqrt{2SI}}$ .

[Pr. 83, let us suppose] it should be, let that superficies be not a purely, &c. as it is in the original.

### SECTION XIII.

[Pr. 90, cor. 1, 2] for the fluxion of the area is as  $D - nD$ , and the area as  $\frac{D^{1-n}}{1-n}$ , or as  $\frac{1}{D^{n-1}}$ ; and the area ALIH as  $\frac{1}{PA^{n-1}} - \frac{1}{PH^{n-1}}$ .

[Pr. 91, cor. 1. And the other part  $\frac{PF}{PR}$ ] for the fluxion of that area is  $= \frac{PF}{PR} \times P\dot{F} = \frac{PF}{\sqrt{RF^2 + PF^2}} \times P\dot{F} = \frac{I}{RF^2 - PF^2} \times PF \times P\dot{F}$ . And the area  $= \frac{I}{RF^2 + PF^2} \times PR$ . And (the area on PB is PE - AD; and on PA, PD - AD. And their difference) PE - PD is the area on AB.

[Pr. 91, cor. 2.] This is calculated in the Appendix, by the help of the quadratures of curves, which see in Harris's Lex., vol. 2.

And the curve KRM is a conic section; for the indetermined quantities rise only to two dimensions. Let  $SA = r$ ,  $SC = c$ ,  $BE = u$ ,  $PB = b$ ,  $ER^2 = PD^2 = PE^2 + ED^2 = \overline{b - u^2} + \frac{cc}{rr} \times \overline{2ru - uu} = bb + \frac{2cc}{r} - 2b \times u + \overline{1 - \frac{cc}{rr} \times uu}$ .

[Sch. Pr. 93, and I suppose the force proportional.] For (Fig. 33) consider A, B, as flowing quantities, and let  $\dot{A}$  be given = o. Now since  $A \frac{m}{n} = B$ ; therefore when A becomes  $A + \dot{A} = A + o$ , B becomes  $B + \dot{B}$ , and then  $B + \dot{B} = \overline{A + o} \frac{m}{n} = A \frac{m}{n} + \frac{m}{n} oA \frac{m-n}{n} + \frac{m^2 - mn}{2n^2} ooA \frac{m-2n}{n} + \&c.$  Now if any quantity  $\overline{A + o} \frac{m}{n}$  (where o is infinitely small) be involved, the first, second, third, &c. term will be respectively as the flowing quantity ( $A \frac{m}{n}$ ), the first, second, &c. fluxion of that quantity, as is demonstrated in the quadrature of curves, sch. to pr. 11 (for which see Harris's Lexicon). Therefore  $\frac{m^2 - mn}{2n^2} ooA \frac{m-2n}{n}$  is as the

second fluxion of  $A \frac{m}{n}$ . Now if a (centrifugal) force act from the line CD in the direction B, and a body move in the curve CZ, the fluxion of A will continue the same always, and the force acting upon the body will be every where as the second fluxion of B, or of  $A \frac{m}{n}$  (its equal), and therefore as  $\frac{m^2 - mn}{2n^2} ooA \frac{m-2n}{n}$ , or (because o is given) as  $\frac{m^2 - mn}{n^2} A \frac{m-2n}{n}$ , or as  $\frac{m^2 - mn}{n^2} B \frac{m-2n}{m}$ . Or  $\frac{mn - m^2}{n^2} B \frac{m-2n}{m}$  is as the centripetal force. This may be more easily

shewn by fluxions only, by finding the second fluxion of (B or)

$$A^{\frac{m}{n}}.$$

## SECTION XIV.

[Pr. 94, equal to the square of HM] (Fig. 34) for imagine a diameter drawn through H (Fig. 35), and an ordinate to it through I, the abscissa will be = MI, and the ordinate = MH. Also, completing the figure, the tangent IP cuts from the diameter Ha, Hp = Hu = MI; therefore the triangles HLP, MLI, are equal and similar, and HL = LM. And the *latus rectum* to Ha is always the same, whatever the angle of incidence be, if the velocity be given; for the line Hd that the body will describe in a given time is given; and the line dg that the body approaches the plane in a given time is

given; and therefore  $\frac{Hd^2}{Hz}$ , or  $\frac{zg^2}{zH}$ , or the *latus rectum*, is given.

[Pr. 96, sch. of the secants] of the angles between the line of incidence and the plane.

[Pr. 97, cor. 2.] The curve lines CP, CQ (being every where perpendicular to AP, DK), are composed of arcs of circles; therefore PD is the increment of AP or AC, and QD the decrement of QK or CK; and therefore these increments are as the sines of incidence and emergence. And, *contra*, if PD, QD, are as the sines of incidence and emergence, a body moving in the line PD shall emerge in the line QDK, by this prop.

[Pr. 98, and therefore QS to be always equal to CE;] (Fig. 36). For since qqqq and ssss are always perpendicular to qs, therefore supposing QS and 1qs to intersect in K, then KQ — KS = K. 1q — KS, that is, qs = 1q.s. Also let 1q.K and 2q.s intersect in T; then 1q.s = 1q.T — ST = 2T — sT = 2q.s. Also 2q.s = 2q.V — SV = 3q.V — SV = 3q.S; and so on, till at last qs, coinciding with CE, will be equal thereto. Therefore QS is every where equal to CE.

[Ib. sch.] See the Author's Optics, where all these things are shewn.

## BOOK II.

## SECTION I.

[Pr. 2, cor. For if that area] by Marq. Hospital's Conic Sect., art. 219; and my Conic Sect., prop. 86, hyperbola, cor. 6.

[Ib. of the right line AC] by the converse of lem. 1; but the contrary does not hold, that is, if any area ABGD be taken for the time, that DC shall represent the velocity, or AD the space.

[Pr. 3. And the resistance] The force of gravity may be compared with the resistance of the medium; for they may both be considered as uniform pressure.

[Ib. or AC to  $\frac{1}{2}AK$ ] For suppose a tangent to be drawn to the point B, then  $Kq : (\text{subtangent } \equiv) AC :: kq : Bk$  or  $AK$ . And in general in any area, as MstN, it will be,  $AC : \frac{1}{2}MN$

$+ NC :: (\text{by the nature of the hyperbola}) \frac{Ms + Nt}{2} : AB$  or  $Mm$ . And by division  $AC : AM + \frac{1}{2}MN :: \frac{Ms + Nt}{2} :$

$\frac{ms + nt}{2} :: \text{area } sMNt : \text{area } smnt$ . But  $AG : AM + \frac{1}{2}MN :: \text{force of gravity} : \text{resistance in the middle of the fourth time}$ . Therefore  $sMNt : smnt :: \text{force of gravity} : \text{resistance in the midst of the fourth time}$ , and so of the rest.

And the like demonstration holds (Fig. 37) in ascending motion; for let the rectangle  $ABDG$  be divided into innumerable rectangles  $Dk, Kl, Lm, Mn, \&c.$  which shall be as the decrements of the velocities produced in so many equal times; then will  $AE, Ak, Al, Am, \&c.$  be as the whole velocities, and therefore as the resistances of the medium in the beginning of each of the equal times. Let  $AC : AK :: \text{force of gravity} : \text{resistance in the beginning of the second time}$ ; then to the force of gravity add the resistances, and  $DEHC, KkHC, LIHC, MmHC, \&c.$  will be as the absolute forces, or  $\propto$  decrements of the velocities,  $\propto Dk, Kl, Lm, Mn, \&c.$  and therefore  $\propto$ ; whence  $DGqK = KqrL = LrsM = MstN, \&c.$  will be  $\propto$  the equal times or forces. But  $AC :$

MC :: Ms : AB. And by division AC : AM :: Ms : ms :: area sMNT : area smnt :: (and therefore as) force of gravity : resistance in the fourth time; and so of the other areas. Therefore since DGqK, KqrL, LrsM, MstN, are  $\propto$  the gravitating forces, the areas GEkq, qklr, rlms, msnt,  $\propto$  resistances in each time,  $\propto$  velocities  $\propto$  spaces described. And, by composition, in the times DGrL, DGBA, the spaces described will be as GElr, GEB. Q.E.D.

[Ib. cor. 3.] For the diff. spaces  $\propto$  velocities (that is, by cor. 2)  $\propto$  CA, CK, CL ( $\therefore$ ), or  $\propto$  AK, KL, LM ( $\therefore$ ), by lem. 1 (Fig. 3).

[Ib. let that also be distinguished] This is true in this law of resistance; because the motion lost is  $\propto$  velocity  $\propto$  remaining velocity  $\propto$  spaces described.

[Pr. 4, which is the locus of the point r. Q.E.D.] all the rest is plain by prop. 2 and 3, and cor. pr. 2.

Ib. cor. 1, that is, if the parallelogram] For DA : CP :: DR : RX =  $\frac{DR \times CP}{DA}$ . But DAB =  $\overline{DC - AC} \times AB$  = (by the nature of the hyperbola)  $\overline{AB - AQ} \times DC = QB \times DC =$  (by construction) N  $\times$  CP. And DA =  $N \times CP$ . Wherefore RX =  $\frac{DR \times CP \times AB}{N \times CP} = \frac{DR \times AB}{N}$ .

[Ib. cor. 4, DraF is also given.] by this prop.

[Ib. cor. 6.] for then  $2DP \propto \frac{\text{lat. rect.}}{\text{resistance}} \propto \frac{\text{lat.}}{\text{velocity}} \propto \frac{\text{velocity}^2}{\text{velocity}} \propto \text{velocity}$ .

[Ib. cor. 7, the ratio  $\frac{Ff}{DF}$ ] For when this ratio is the same with the other, the curves described (and supposed to be described) in these two cases will be similar.

## SECTION II.

Pr. 5 had been better expressed thus (though the translation agrees with the original).

*If a body is resisted in the duplicate ratio of its velocity, and moves by its innate force only through a similar medium,*

*and the spaces be taken equal, I say, that the times of their description are in a geometrical progression increasing; and that the velocities at the beginning of each of the times are in the same geometrical progression (decreasing or) inversely.*

For it is plain that the times may be taken in such a geometrical progression as that the spaces cannot be equal, nor the velocities in the same inverse progression. But if the spaces are equal, the times and velocities will be in a geometrical progression, inverse to one another. The demonstration is the same, however the proposition be expressed; for it being proved that the lines AB, Kk, Ll, &c. being squared, the squares are as the differences of the lines; and also that the squares of the velocities are as the differences of the velocities. Therefore if AB and Kk be taken as the velocities in the beginning of the times AK, KL, and an hyperbola be drawn through the points B, k, to the asymptote CD (whose centre let be C), all the other velocities will be as the lines Ll, Mm, &c. because the progression, of the lines as well as the velocities continues the same all along. Whence the spaces (in these equal times) will be as the areas Ak, Kl, Lm, &c. and in the time AM the space will be as AMmB. Now conceive the area AMmB to be divided into the equal areas Ak, Kl, Lm, &c. Then will CA, CK, CL, &c. be  $\propto$  increasing; and the parts (by lem. 1) AK, KL, LM, &c. which are as the times, will be in the same progression, increasing. Also the velocities AB, Kk, Ll, &c. (which are reciprocally as CA, CK, CL, &c.) will be in the same progression inversely. Q.E.D.

[Pr. 7, and times conjunctly] very small particles of time.

[Ib. they will always describe spaces,] for the parts of space described in those several particles of time are as the respective velocities  $\times$  those several parts of time; that is, as the first velocity  $\times$  each part of time. And, by composition, the whole spaces are as the first velocities  $\times$  whole times. Q.E.D.

[Pr. 7, cor. 3.] dele, Let those diameters applied to that power.

[Pr. 8.] in Fig. 1, Pl. 2, k ought to be beyond l in respect of A, &c. because the resistance decreases. But the demonstration is the same any way. But this figure is not in the 1st or 2d edit. of the original. In prop. 8 and 9, AC represents the relative gravity, or its weight in the fluid.

[Pr. 9, cor. 7.] For the greatest velocity is given (by pr. 8, cor. 2, 3). And thence the time of acquiring that velocity in free space. And ABNK (ABnk) being given, there is given AK, and AP (Ap), and ATD (AtD). Then  $ADC : ADT$  ( $ADt$ ) :: time of acquiring that greatest velocity in free space : time sought.

[Pr. 10, equal among themselves] and very small.

[Ib. or  $\frac{MI \times NI}{HI}$ ] as appears by similar  $\Delta s$ , produced by letting fall a  $\perp$  from N on HI.

If the velocity in GH be greater than the velocity in HI, the decrement is  $\frac{GH}{T} - \frac{HI}{t}$ , and arises from the resistance and gravity together (because gravity draws the body from the tangent into the arc HI); and if gravity act not, that decrement would be greater (for gravity accelerates the motion) by  $\frac{2MI \times NI}{t \times HI}$ ; therefore the decrement by the resistance alone is  $\frac{GH}{T} - \frac{HI}{t} + \frac{2MI \times NI}{t \times NI}$ .

[Ib. will be NI;] for Q being a given (ratio or) quantity,  $Qo$  will always represent MN, whatever the magnitude of o be.

And the ordinate  $DI = CH - MI$ ; also the value of MI in the ordinate EK is (because o becomes 2.o, and substituting  $2 \times o$  for o)  $2Qo + 4Ro^2 + 8Ro^3 + \&c.$ ; and therefore  $EK = CH - 2Qo - 4Ro^2 - 8So^3 - \&c.$  Also in the ordinate BG, when o becomes — o, the value of MI (by substituting — o for o) will be —  $Qo + Ro^2 - So^3 + \&c.$  And  $BG = CH - MI$  (as before)  $= P + Qo - Ro^2 + So^3 - \&c.$

[Ib. or  $\frac{R + \frac{3}{2}So}{R}$ ] for (because  $o$  is infinitely small)  $R$ ,  $R + \frac{3}{2}So$ ,  $R + 3So$ , are both arithmetical and geometrical proportionals.

[Ib. by substituting the values — ] becomes  $1 + \frac{3So}{2R} \times$

$$o\sqrt{1+QQ} - \frac{Q Roo}{\sqrt{1+QQ}} = o\sqrt{1+QQ} - \frac{Q Roo}{\sqrt{1+QQ}} \\ + \frac{2Qo + 2Ro^2 \times Ro^2 + So^3}{o\sqrt{1+QQ} + \frac{Q Roo}{\sqrt{1+QQ}}} = (\text{neglecting the superfluous powers of } o, \text{ or the quantities infinitely less than the other}) o$$

$$\sqrt{1+QQ} - \frac{Q Roo}{\sqrt{1+QQ}} + \frac{3So^2}{2R} \sqrt{1+QQ} = o\sqrt{1+QQ} \\ - \frac{QRo^2}{\sqrt{1+QQ}} + \frac{2QRoo}{\sqrt{1+QQ}} = \frac{3So}{2R} \sqrt{1+QQ}.$$

[Ib. *latus rectum*  $\frac{HN^2}{NI}$ ] or  $\frac{HI^2}{NI} = (\text{neglecting the superfluous powers of } o)$   $\frac{o \times \sqrt{1+QQ}}{Roo} = \frac{1+QQ}{R}$ .

[Ib. p. 33, that a body by ascending from  $P$ ] or any other point of the quadrant  $PF$ , in the direction of the arc of that quadrant.

[Ib. Ex. 3, the second term  $\frac{m}{n}o - \frac{bb}{aa}o$ ] or rather  $\frac{bb}{aa}o - \frac{m}{n}o = Qo$ ; but the square is the same either way.

[Ib. Ex. 4, for  $Qo$ ]  $\frac{nbb}{A^n + 1}o - \frac{d}{e}o = Qo$ .

[Ib. Sch. And therefore if a curve] for the quantity  $\frac{s}{R^{\frac{4-n}{2}}} \div \frac{HT^{n-1}}{AC}$  is as the density, and if that be given the density is given.

[Ib. than these hyperbolas here described.] For let the two mediums be of the same density in the vertex of the curves; then if the bodies be projected from A, the uniform medium being more dense, and the other less dense at A, the body moving in the uniform medium will be more impeded, and consequently will descend the more than that describing the hyperbola, which is less resisted, and which therefore moves nearer a straight line: therefore the body moving in the uniform medium is more distant from the asymptotes in the vertex of the figure than the other body. But if the densities of the mediums were supposed equal in A, then (by the same way of reasoning) the body in the uniform medium would be nearer the asymptotes in the vertex. And, generally speaking, these curves cannot therefore differ very far.

[Ib. will touch the hyperbola in G,] for suppose the tangent TG to intersect MX. Then (by fluxions) the distance between that point of intersection and V is  $= \frac{VX}{n}$ . Then by

$\text{sim. } \Delta s \frac{VX}{n} : VG :: VX : TX - VG = nVG$ . But  $VY = nVG$ . Therefore  $TX = GY$ .

[Ib. and the velocity] for the velocity (when the gravity is given) is as the ordinate of the parabola, that is (the abscissa being given, by the gravity), as the  $\sqrt{latus\ rectum}$ .

[Ib. Rule 1.] For the velocities are as the  $\sqrt{s}$  of the *latus rectum* of two parabolas, that is, as  $\sqrt{\frac{2XY^2}{nn+n}} \times VG$ , that is, as  $\sqrt{\frac{2AH^2}{AI}}$  to  $\sqrt{\frac{2Ah^2}{Ai}}$ , which (by supposition) are in a ratio of equality, and therefore  $\frac{AH}{AI} = \frac{Ah}{Ai}$ . Also since the densities are equal,  $\frac{n+2}{3XY}$  will be given, that is,  $\frac{n+2}{3AH} = \frac{n+2}{3Ah}$ .

Whence  $AH = Ah$ . Whence  $AH, AI$ , remain the same; and  $HX$ , which is composed of  $AI$  and the subtangent to the axis  $XH$ , is the same also.

[Rule 2.] For velocity  $\propto \sqrt{\frac{AH^2}{AI}} \propto$  (because AH is given)

$$\checkmark \frac{1}{AI}.$$

[Rule 3, and therefore AH] for let b be the ratio, then  
 $\frac{Ah}{Ai} = b \frac{AH}{AI}$ . And  $\frac{AH^2}{AI} = \frac{Ah^2}{Ai} = b \frac{AH}{AI} \times Ah$ . Whence  
 $Ah = \frac{AH}{b}$ ; also,  $\frac{Ah}{Ai} = \frac{AH}{bAi} = b \times \frac{AH}{AI}$ . Whence  $Ai = \frac{AI}{bb}$ .

[Rule 4, a little greater] since AH decreases faster than GT, therefore the sum of all or every AH + GT is lesser than the first AH + GT  $\times$  number of them; and therefore the sum of all the densities (being reciprocal thereto) is greater than the sum of as many densities in A and G. But the mean density is = sum of all the densities  $\div$  by the number of them, and therefore is something greater than the  $\frac{1}{2}$  sum of the densities in A and G. But half sum densities :

density in A ::  $\frac{AH + GT}{2}$  : GT. Therefore mean density :

density in A :: is in a ratio a little greater than  $\frac{1}{2} \times \overline{AH + GT}$  : GT.

[Ib. Rule 5.] for HX = AI + subtangent = AI + nAI, XN being the axis.

[Ib. Rule 6, by how much] for the variation of curvature in one of these hyperbolas, where n is great, is least in the part AG, and so it is in the curve of projection, where the velocity is greatest; so they agree in that part. But they differ more in the part GK, where the projectile approaches to a uniform motion.

[Ib. Rule 7.] AH to 2AI ought to be 3AH to 4AI (but the original is AH to 2AI).

[Ib. Rule 8, whose conjugate] hyperbola shall pass through the point C.

[Ib. what has been said] for by supposing the index n to be negative, VG (which before was as  $VX - n$ ) will then be as

VX<sup>n</sup>, and the curve AG will be a parabola; and what was demonstrated generally of the hyperbola will hold true (for any index, and therefore) for the parabola; and all things will follow as before, substituting — n for n. But the computation may be made for the parabola as for the other (Fig. 38) thus: let VG = bbXV<sup>n</sup> = XC<sup>n</sup>; then the subtangent ZT = nVG. And let XC = a, ng = e, cC = o; then VG =  $\frac{a + o^n}{a + o^n} = a^n + na^{n-1}o + n \times \frac{n-1}{2} a^{n-2} oo + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} o^3 = P - Qo - Ro - So^3$ .

Whence Q = — na<sup>n-1</sup>. R = — n ×  $\frac{n-1}{2} a^{n-2}$ . S = — n ×  $\frac{n-1}{2} \times \frac{n-2}{3} \times a^{n-3}$ . Therefore the density in G is as  $(\frac{S}{R\sqrt{1+QQ}} = ) \frac{n-2}{3a\sqrt{1+n^2a^{2n-2}}}$ ; that is, as  $\frac{1}{\sqrt{a^2+nna^{2n}}}$ , or as  $\frac{1}{\sqrt{a^2+nne}}$ ; that is, reciprocally as  $\sqrt{XC^2 + TZ^2}$ . And the same may be found by fluxions, putting Q, R, S, for the first, second, third, fluxion of VG or XC<sup>n</sup>.

### SECTION III.

[Pr. 12, and inversely as the velocity;] for the time of describing a given space is reciprocally as the velocity.

[Pr. 13, case 1, 2, 3, the decrement or increment PQ] in a given time.

[Ib. cor. and triangle do.] therefore, *ex equo*, &c.

[Ib. sch.] For (Fig. 39) draw the indefinite line BAP, and make BD perp. and equal to BA; and draw DF, AF, parallel to BA, BD. Let AP be the velocity, AP<sup>2</sup> + 2BAP the resistance, AB<sup>2</sup> the force of gravity. Draw DTP, cutting FA in T, and the time of the whole ascent will be as the triangle DTA.

For draw DVQ cutting off PQ the moment of velocity, and DTV the moment of the triangle; then the decrement of the velocity PQ will be as the resistance and gravity AP<sup>2</sup>

$\dagger 2BAP + AB^2$ ; that is, as  $BP^2$ . But the area DTV is to the area DPQ as  $DT^2$  to  $DP^2$ , or as  $DF^2$  to  $BP^2$ ; therefore since the area DPQ is as  $BP^2$ , the area DTV will be as the given quantity  $DF^2$ . Therefore the area ADT decreases uniformly as the time, by the subduction of given particles DTV; and therefore is proportional to the whole time of ascent.

[Pr. 14. From the area DET] This should be:—from the moment KLON subduct DTV, or  $m \times BD$ , the moment of DET; and is only a false translation.

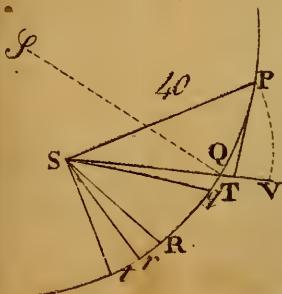
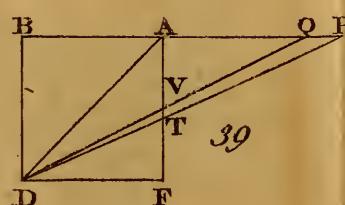
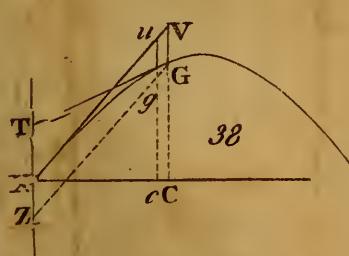
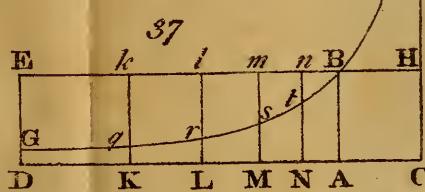
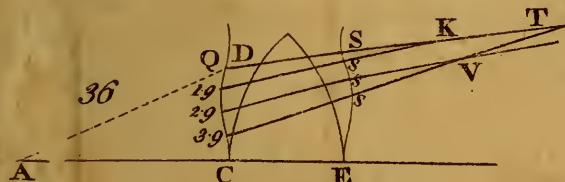
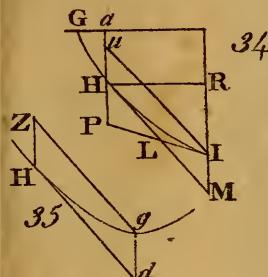
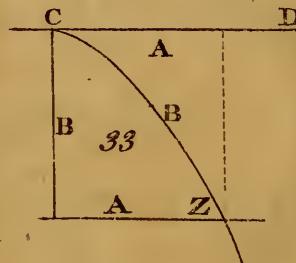
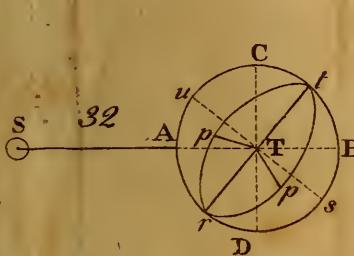
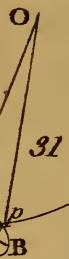
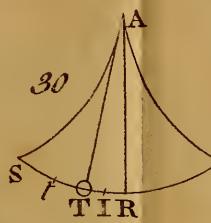
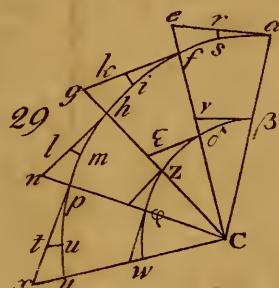
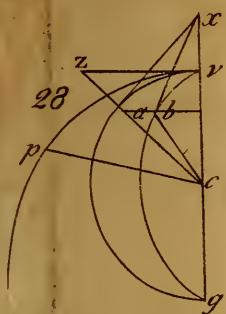
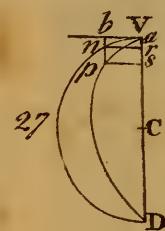
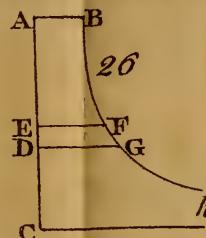
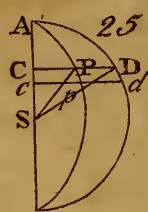
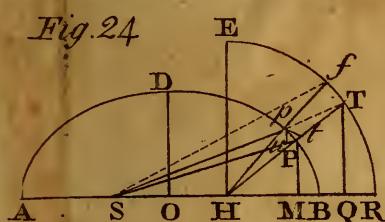
[Ib. cor. or as  $V^2$ ;] for the space is  $\propto$   $\square$  of the time, that is,  $\propto$  the square of DET, or of  $BD \times M \propto$  (because  $M = \frac{DE \times V}{DA}$ ) square of  $\frac{BD \times DE \times V}{DA}$ , that is (because  $BD$ ,  $DE$ ,  $DA$ , are given), as  $V^2$ , or as  $\frac{BD \times V^2}{AB}$ .

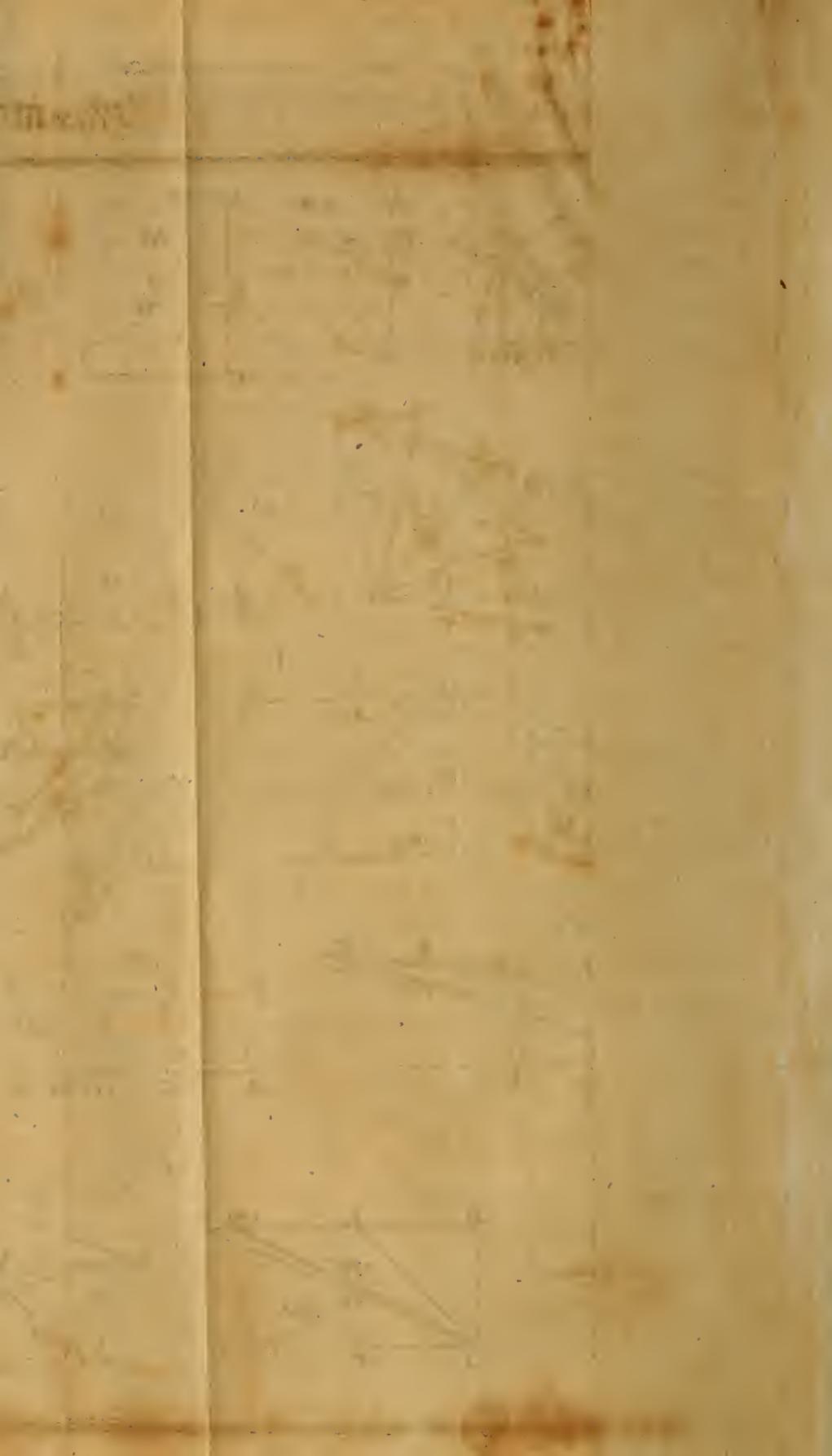
[Ib. sch. instead of the uniform resistance made to an ascending body;] this differs from gravity only in this, that it cannot generate any motion; but it acts after the same manner in all moving bodies in destroying their motion as gravity does in destroying the motion of ascending bodies. Gravity acts uniformly in a given direction. The force arising from tenacity acts uniformly, but always in a direction contrary to the motion of the body; and, therefore, when the body is at rest, it can induce no change in it. Now in the horizontal motion of a body in a fluid which is resisted in part uniformly, one may substitute the force of gravity for that uniform resistance, as in pr. 8, 9, 13, 14. And in the ascent or descent of the body in a fluid, instead of the force of gravity, one may substitute the sum or difference of that uniform force and the force of gravity, as in these propositions.

#### SECTION IV.

In prop. 15. [And the decrements of these arcs arising from the resistance will be as the squares of the times in which they are generated;] for since the arcs are infinitely small, the resistance affects the same as an uniform centripetal force would do.

*Fig. 24*





b. The decrement of the arc (Fig. 40) PQ will be] for let q, t, be the points it would arrive at, in the same times, in free space; then  $Pq = qt$ , and  $PQ = Qt$ ; therefore  $Qq = qt - Qt$ ; and adding  $Qq$ ,  $2Qq = Qt - Qt = tr$ ; and  $Qq = \frac{1}{2}tr$ : but  $Qq = \frac{1}{4}Rt$ ; therefore  $tr = \frac{1}{2}Rt = Rr$ , and  $Qq = \frac{1}{2}Rr$ .

The same demonstration of this prop. holds good as well in ascending as in descending motion.

[Cor 1. The velocity, &c.] for supposing PQ an arc of a circle (Fig. 41) whose centre is f; then if PQ or PT be as the velocity in both curves, TQ will be as the centripetal force in both, but in both curves  $TQ = \frac{\overline{PQ}^2}{2\overline{Pf}}$ ; therefore  $Pf = \text{rad.}$  of the circle described with the same velocity.

[Cor. 4, the body will descend, &c.] for, by cor. 1, the velocity in P in a resisting medium is  $=$  velocity in a circle at distance SP in free space; but velocity at distance SP in free space : to velocity at dist.  $\frac{1}{2}SP$  in free space (by cor. 6, prop. 4, b. I.), or prop. 34, :: as  $\sqrt{\frac{1}{SP}} : \text{to } \sqrt{\frac{1}{\frac{1}{2}SP}}$  ::, that is,  $\sqrt{1} : \sqrt{2}$ ; therefore, &c.

[Cor. 7, or as  $\frac{2}{3}AS$  to AB] (for this, see my Geometrical Proportion, prop. 24.)

In prop. 16. [And the resistance, &c.] instead of this proportion in prop. 15, viz.  $PQ : QR :: SQ : \sqrt{SP \times SQ}$ , substitute this;  $PQ : QR :: (\overline{SQ^2} : \overline{SP^2} ::) \overline{SQ} : SQ + \frac{n}{2}SP - \frac{n}{2}SQ$  (by prop. 24 of my Geomet. Proportion), that is, as the describing velocities, &c.

[Cor. 1.] For the resistance : centripetal force :: ( $\frac{1}{2}Rr =$ )  $\frac{1 - \frac{1}{2}n \times VQ \times PQ}{2SQ} : (TQ = ) \frac{\frac{1}{2}PQ^2}{Sp} : : 1 - \frac{1}{2}n VQ ; PQ ::$   
 $I - \frac{1}{2}n OS : OP.$

## SECTION V.

In prop. 19. [Case 3, different spherical parts have equal pressures,] that is, different spherical parts of equal magnitude.

In prop. 20. [And by a like reasoning, &c.] Supposing the thicknesses of the orbs to be reciprocally as the force of gravity or the density of the fluid, or in the complicate ratio of them, and it will appear (as before) that the bottom is pressed with a cylinder of the same fluid whose base is = the bottom, and height that of the fluid, by the same reasoning as before, and prop. 19.

In schol. pr. 22. [By a like reasoning, &c.] Let the force of gravity be reciprocally as the  $n$ th power of the distance.

The specific gravities at A, B, C, &c. will  $\propto \frac{AH}{SA^n}, \frac{BI}{SB^n}, \frac{CK}{SC^n}$ , &c. and the densities  $\propto$  sums of the pressures  $\propto \frac{AH \times AB}{SA^n}$ ,

$\frac{BI \times BC}{SB^n}, \frac{CK \times CD}{SC^n}$ , &c.  $\propto \frac{AH}{SA^{n-1}}, \frac{BI}{SB^{n-1}}, \frac{CK}{SC^{n-1}}$ ,

&c. And tu, uw, &c.  $\propto$  differences of the densities  $\propto \frac{AH}{SA^{n-1}}, \frac{BI}{SB^{n-1}}$ , &c. And tu  $\times$  th, uw  $\times$  ui  $\propto \frac{AH \times th}{SA^{n-1}}$ ,

$\frac{BI \times ui}{SB^{n-1}} \propto \frac{1}{SA^{n-1}}, \frac{1}{SB^{n-1}}, \propto \frac{1}{SA^{n-1}} - \frac{1}{SB^{n-1}}, \frac{1}{SB^{n-1}}$  —  $\frac{1}{SC^{n-1}}$ , &c.; and therefore, universally,  $\frac{1}{SA^{n-1}} - \frac{1}{SE^{n-1}} \propto$  hyperbolic area thmy. Therefore if  $\frac{SA^n}{SA^{n-1}}, \frac{SA^n}{SB^{n-1}}$ ,

$\frac{SA^n}{SC^{n-1}}$ , or  $\frac{1}{SA^{n-1}}, \frac{1}{SB^{n-1}}, \frac{1}{SC^{n-1}}$ , be  $\ddot{\propto}$ , then  $\frac{1}{SA^{n-1}} - \frac{1}{SB^{n-1}} = \frac{1}{SB^{n-1}} - \frac{1}{SC^{n-1}}$ , and the areas proportional thereto will be equal, viz. thiu = uikw; and therefore St, Su, Sw, that is, AH, BI, CK  $\ddot{\propto}$ . Q.E.D. Then putting n successively = 3, 4, 0 — 1, &c. the truth of the said scholium will appear. And so on *ad infinitum*.

[Other laws of condensation, &c.] Suppose the compressing force  $\propto$  rth power of the density; then the specific gravities in A, B, &c.  $\propto \frac{AH}{SA^2}, \frac{BI}{SB^2}$ , &c.; and the pressures in A,

B, &c.  $\propto \frac{AH}{SA} + \frac{BI}{SB} + \frac{CK}{SC} + \frac{DL}{SD}$ , &c.  $\frac{BI}{SB} + \frac{CK}{SC} + \frac{DL}{SD}$ ,

&c. Now if these sums were  $\ddot{\approx}$ , their differences  $\frac{AH}{SA}, \frac{BI}{SB},$   
 &c. would be in the same  $\ddot{\approx}$ , and then the pressures in A, B,  
 &c.  $\propto \frac{AH}{SA}, \frac{BI}{SB}$ , &c.  $\propto \text{densities}^r \propto AH^r, BI^r, \text{&c.}$  And  
 therefore  $AH^{r-1}, BI^{r-1} \propto \frac{1}{SA}, \frac{1}{SB};$  and  $AH, BI \propto$   
 $\frac{1}{SA^{\frac{1}{r-1}}}, \frac{1}{SB^{\frac{1}{r-1}}}.$  Then putting  $r = \frac{4}{3}, \frac{5}{3}, 2, \text{ &c.}$  all things  
 will appear as in this schol. But now the aforesaid sums are  
 not in geometrical progression, because it is plain their dif-  
 ferences are not, viz.  $\frac{AH}{SA}, \frac{BI}{SB}, \text{ &c.};$  for if  $\frac{AH}{SA}, \frac{BI}{SB}, \text{ &c.} \ddot{\approx},$   
 and seeing  $SA, SB, \text{ &c.}$  are  $\ddot{\approx}$ , therefore (by pr. 18, cor. 2, of  
 my Geom. Proportion)  $AH, BI, \text{ &c.}$  would be  $\ddot{\approx}$ , which is  
 false (except in the case of prop. 21); therefore neither these  
 differences nor sums are  $\ddot{\approx}$ . And consequently the conclu-  
 sions of this schol. in this respect are not true; and therefore  
 it must be presumed that the author meant them only to be  
 nearly, and not perfectly, true.

In Schol. prop. 23. [All these things are, &c.] It seems to  
 me that the prop. holds true, though the centrifugal force be  
 extended to all distances in any given law, as well as when it  
 ends at the next particle in the same given law. For, sup-  
 posing the particles of the fluid to be placed in parallel planes,  
 and that these planes act on each other, the force of com-  
 pression on any one plane will be made up of the several  
 forces of all the other equidistant parallel planes. Therefore  
 since the distances of all these planes before compression are  
 respectively proportional to their distances after compression  
 from this given plane, their actions on the given plane before  
 and after compression will also be proportional (seeing their  
 actions or forces are as some power of the distances); and  
 their sums will be also proportional to those of any two cor-  
 responding planes, viz. as the nearest to the nearest. In these  
 sort of particles there will be required a greater force to pro-  
 duce an equal condensation of an equal quantity of the fluid;

but the density will be proportional to the compression, as before.

### SECTION VI.

In prop. 24. Cor. 5. — for let  $m, M =$  quantities of matter,  $w, W =$  weights,  $t, T =$  times, for the common pendulum 1. Also  $L =$  new pendulum,  $T =$  new time, for  $M, W$ . Then  $m : M :: wtt : WTT$  (by the prop.) ::  $\frac{wtt}{t} : \left( \frac{wtt}{t} = \right) \frac{WTT}{L}$ . And if  $m, M; t, T$ , are equal,  $w \propto 1$ , which is cor. 4.

In prop. 27. [If the resistance in the arc  $B$ , &c.] If the resistance in the arc  $A$  were to the resistance in the arc  $B$  as  $AA$  to  $AB$ , the times would be equal; and therefore resistance  $AA$  in the arc  $A$  causes the excess of time above that in a non-resisting medium; and resistance  $AB$  in the arc  $B$  causes the excess of time in  $B$  above that in a non-resisting medium, equal to the former excess (because with these resistances), they are described in equal time. But now if  $B$  be described with a resistance  $BB$ , the excess of time will be caused by that resistance.

[But those excesses, &c.] Now the excesses of time caused by describing  $B$  with the resistances  $AB, BB$ , respectively are as the spaces left undescribed after the given time; that is, as the resistances  $AB, BB$ , nearly, or as  $A$  to  $B$ , viz. as the arcs  $A, B$ . And since the excess of time by describing  $A$  with resistance  $AA =$  excess of time by describing  $B$  with resistance  $AB$ , therefore it follows that excess of time in  $A$  : to excess of time in  $B$  :: as  $A$  : to  $B$  nearly.

In prop. 29. [Area  $PIEQ$  may be to the hyperbolic area  $PITS ::$  as  $BC$  to  $Ca$ ; and that the area  $IEF$  may be to the area  $ILT$  as  $OQ$  to  $OS$ .]

That this is possible (assuming  $P$  at pleasure) may be thus shewn. Let the areas  $PIEQ, PITS$ , be taken (as  $BC$  to  $Ca$ ) indefinitely small; then  $PIEQ$  must be greater or at least equal to  $PITS$ : suppose them equal; then will  $PIEQ > PILS$ , and  $IF > IL$  (and more so, if  $PIEQ > PITS$ ); but when  $S, P, Q$ , coincide,  $OS$  to  $OQ$  is a ratio of equality; therefore in this

case IFE to ILT (or IF to 1L) > OQ to OS (or however not less) increase the areas PIEQ and PITS in the same ratio, and the ratio of OQ to OS will also increase and converge at length to the ratio of IEF to ILT ; which shews the possibility of what was required (but as to an actual solution I shall refer it to an algebraic process). This being done as required, then the points O, S, P, Q, are all fixed, and the point R only variable.

[And the increment, &c.] for the fluxion of  $\frac{IEF}{OQ}$  OR — IGH to the fluxion of — PIGR :: as  $\frac{IEF}{OQ}$  — HG to — RG :: or as HG —  $\frac{IEF}{OQ}$  to RG. Or rather thus : the increment of  $\frac{IEF}{OQ}$  OR — IGH is as  $\frac{IET}{OQ}$  — HG  $\times$  — Rr = HG  $\times$  Rr —  $\frac{IEF}{OQ}$   $\times$  Rr. And the increment of PIGR is as RG  $\times$  — Rr ; and therefore the decrement (because the decrement is a negative increment) of PIGR is as RG  $\times$  Rr.

In cor. to pr. 30. [Now if the resistance DK be in the duplicate, &c.] for the velocities being as the ordinates of a circle or ellipsis described on aB, the resistances (being as the squares of these velocities) are as the squares of these ordinates, that is, as the rectangles of the abscissas. Therefore the resistances DK, OV, are respectively as aDB (aOB = ), OB<sup>2</sup>. But this is the property of the parabola aVB described to the axis VO, by cor. 2 to pr. 4 of the parabola.

Gen. sch. pr. 31, p. 75. [But in lesser oscillations somewhat greater, &c.] that is, the difference in the lesser arc to the difference in the greater arc is in a greater ratio than the squares of these lesser and greater arcs.

[P. 75.] for  $\frac{1}{2}$  the chords of these arcs, read the chords of those half arcs.

[Ib. p. 76, 77. As  $\frac{7}{11}$  AV +  $\frac{7}{10}$  BV $\frac{3}{2}$  +  $\frac{3}{4}$  CV<sup>2</sup>.] These

three members are taken to denote three laws of resistance, viz. the first in the simple ratio of the velocity, the third in

the duplicate ratio thereof, and the second partly in the simple, partly in the duplicate ratio (or in a mean between them); and  $\frac{7}{16}$  the coefficient thereof is a middle one between  $\frac{7}{11}$  and  $\frac{3}{4}$ .

[As 0,041748 to 121] it should be, as 0,041778 to 121.

[P. 78. It is manifest that the force, &c.] for the forces of resistance and gravity are as their effects; to wit, as the velocity lost (by resistance) and velocity gained (by gravity) in the same time as 1 to  $376\frac{1}{50}$ .

[I also counted—I leave the calculation, &c.] The calculation after the preceding manner will be as follows:

|                                                                      |                 |                 |                 |                  |                  |                 |     |
|----------------------------------------------------------------------|-----------------|-----------------|-----------------|------------------|------------------|-----------------|-----|
| Mean oscillations                                                    | - - - - -       | $\frac{31}{2}$  | 7               | 14               | 28               | 56              | 112 |
| Difference between first descent and last ascent                     | $\frac{1}{2}$   | 1               | 2               | 4                | 8                | 16              |     |
| Difference of arcs described in the descent<br>and subsequent ascent | $\frac{1}{748}$ | $\frac{1}{272}$ | $\frac{4}{325}$ | $\frac{12}{250}$ | $\frac{24}{125}$ | $\frac{48}{68}$ |     |

Also these in the greater arcs are nearly in the duplicate ratio of the velocities, but in the lesser arcs somewhat greater (these arcs are in the last series of the table).

Let  $V$  = velocity in second, fourth, and sixth cases = 1, 4, 16, respectively; and the difference of the arcs will be, in the

$$2d \text{ Case} = \frac{1}{272} = A + B + C.$$

$$4th \text{ Case} = \frac{1}{250} = \frac{6}{125} = 4A + 8B + 16C.$$

$$6th \text{ Case} = \frac{1}{68} = \frac{12}{17} = 16A + 64B + 256C.$$

Whence  $A = ,0005098$ ;  $B = ,0005882$ ;  $C = ,0025784$ . Therefore the difference of the arcs is as  $,0005098V + ,0005882V\frac{3}{2} + ,0025784V^2$ . And consequently the resistance of the globe in the middle of the arc will be to its weight (by cor. pr. 30) as  $,00032442V + ,00041174V\frac{3}{2} + ,0019338V^2$  to 121. Therefore the resistance will be to the weight, in case 2d, 4th, and 6th, as  $,00267$  to 121;  $,0355324$  to 121;  $,5265949$  to 121, respectively.

Note—You may take any other numbers (that are in the same proportion with these above) for  $V$ ; for example, I took  $\frac{1}{4}$ , 1, and 4 =  $V$ , in the 2d, 4th, and 6th cases, and  $A$  will be = ,002039;  $B$  = ,004706;  $C$  = ,041255; and the resistance to the weight (at the last) comes out the same as before,

In the 6th case, the point marked in the thread described an arc of  $112 - \frac{4}{6}\frac{8}{9}$  inches =  $111\frac{5}{17}$ ; therefore the centre of the globe describes an arc =  $115\frac{6}{17}\frac{7}{9}$  =  $115\frac{8}{9}$  nearly; and its velocity is (nearly) the same as in descending an arc  $57\frac{17}{18}$  of a cycloid (whose semi-arc is 126), or = velocity acquired by falling perpendicularly through the versed sine (or abscissa) of that arc  $57\frac{17}{18}$ : but this versed sine is = 13,324; therefore the velocity of the pendulum is = velocity acquired by falling perpendicularly 13,324 inches. And with this velocity the globe meets with a resistance which is to its weight as ,52659 to 121, or (taking that part only of the resistance which is as the square of the velocity) as ,49505 to 121.

Also if a globe of water of equal magnitude moves with the same velocity, its resistance will be to its weight as ,49505 to 213,4; or as 1 to  $431\frac{1}{5}$ . Whence, in the time in which the globe uniformly describes 26,648 inches, the weight of the globe of water will generate all that velocity in the falling globe; therefore the velocity destroyed by resistance will be to that acquired by gravity (in the same time) as 1 to  $431\frac{1}{5}$ , or velocity lost =  $\frac{1}{431\frac{1}{5}}$  of the whole velocity. And therefore, in the time it would uniformly describe its semi-diameter with the same velocity, it would lose the  $\frac{1}{3341.7}$  part of its motion.

[P. 79, after five oscillations] this certainly should be after 10 oscillations, as appears by the process of the calculation.

[P. 80. So that the difference 0,4475, &c.] for the motion lost is  $\propto$  the resistance, and that  $\propto$  the square of the velocity and square of the time nearly, which in this case is a constant product.

[P. 81, l. 2.] Here is a small error, viz. ,61675 instead of ,61705; whence 45,453 would have been 45,43, and the resistances as 7,002 to 1: but this is not material.

[P. 82, because the resistance, &c.] for the resistance  $\propto$  square of the time  $\times$  square of the velocity, which product is invariable, because equal arcs are always described.

## SECTION VII.

[Pr. 32, ratio of density] he means any given ratio of density which are proportional to the particles.

[Pr. 32. Proportional times] any times, in that constant ratio ; that is, let them move among one another in similar directions, with velocities which are as the particles.

[Pr. 32, cor. 2, spaces proportional to their diameters] in these proportional times.

[Pr. 33, cor. 2, for if the forces, &c.] for the resistance arises from the centrifugal forces, and from the collisions of the particles. The resistances of the first sort are as the motive forces, that is, as the accelerative forces and quantity of matter ; that is (by supposition), as the squares of the velocities and quantities of matter ; that is (because the quantities of matter are given, the fluid being the same), as the squares of the velocities.

The resistances of the second sort are as the number of reflexions and their forces ; that is (as is proved in the prop.), as the squares of the velocities, squares of the diameters, and densities of the parts ; that is (because the diameters and densities are given), as the squares of the velocities accurately.

[As also the bodies E and G ;] let E and G be vastly swifter than D and F.

[Pr. 34, but the former of, &c.] for  $bH \times CB = (BE^2 =) ObN$ , which is the property of the parabola.

[Pr. 34, schol. less resisted than the former solid ;] for if FG and IH be produced till they intersect, they will form a right angle ; and therefore the frustum FGBHI will be less resisted than if the lines should intersect in an obtuse angle (as is shewn in the cone), which they would do if the lines fell within FG and IH ; and for the same reason the resistance would be greater in the curve itself, because the lineola which constitute it will, when produced, intersect one another at obtuse angles. What is mentioned besides in this schol. is demonstrated in the Appendix.

[Pr. 35, it follows, that the cylinder, &c.] for motion cylinder : motion medium :: cylinder's magnitude  $\times$  density  $\times$  velocity : (medium's magnitude  $\times$  density  $\times$  velocity =)

cylinder's  $\frac{\text{magnitude}}{2} \times 2 \text{ velocity} \times \text{medium's density} :: \text{cylinder's density} : \text{medium's density}$ .

[Ib. to the force by which] to the motive force by which its whole motion, &c.

[Pr. 35, cor. 2] for the resistance of the globe and the force (that will quite take away its motion in the time it moves two or four thirds of its diameter) are in a given ratio; therefore increase the velocity of the globe in any ratio, and the force (that will utterly destroy all its motion in a given space) will be increased in a ratio of the velocity directly and the time inversely, that is, as the square of the velocity; therefore the resistance of the globe is increased in the duplicate ratio of the velocity.

[Ib, cor. 3] for the same reason, increase the diameter of the globe in any ratio, and the motive force (that will utterly destroy its motion) will be increased in the triplicate ratio of the diameter directly, and the time inversely; that is, in the triplicate ratio of the diameter directly, and the simple ratio of the diameter inversely; that is, as the square of the diameter; and therefore the resistance of the globe will be increased in the duplicate ratio of the diameter.

[Cor. 6, and its resistance, &c.] Let BC be the resistance at first, then resistance at the first : resistance at the end ::  $BC^2 : EF^2$ ; but  $AE : AB :: BC : EF :: EF : BH$ ; therefore  $BC^2 : EF^2 :: BC : BH$ . ∴ BH is the resistance at the end.

[Cor. 7. As the logarithm, &c.] For (by schol. prop. 86, Fig. 42, hyperbola) it appears that if  $AH = 1$ ,  $AB = T$ ,  $BK = t$ ; then it will be  $CBKF : EHKF :: (\logarithm T + t - \logarithm T) = \logarithm \frac{T+t}{T} : \logarithm \overline{T+t}$ . And

$EHKF : AHED :: (\logarithm \overline{T+t} : ,43429448 :: ) \logarithm \overline{T+t} \times 2,30258509 : 1$ ; also  $AHED : CBKG :: HE : CB \times t :: T : t$ . Therefore, *ex equo*,  $CBKF : CBKG :: \logarithm \frac{T+t}{T} \times 2,30258 \times T : t :: \logarithm \frac{T+t}{T} \times 2,30258509 : \frac{t}{T}$ .

[Pr. 36, cor. 2.] Forces are equal when their effects are equal in a given time; but the effect of the motion of the effluent water (in the time a body descends through GI) = a cylinder whose length is  $2GI$ , and base the hole EF; also the effect of the gravity or weight of a column whose height is  $GI$ , and base the hole EF, = (by reason of an accelerated motion, and in the same time) a cylinder whose length is  $GI$ , and base the hole EF; and therefore the effect of the pressure of twice that column (in the same time) = a cylinder whose length is  $2GI$ , and base the hole EF; = the first cylinder, *ergo*, &c.

[Cor. 3, or  $IH + IO$  to  $2IH$ ] for  $ih : ho :: io : og$ ; or  $ho : og :: ih : io$ ; and  $ho + og : 2ho :: ih + io : 2ih :: io + ig : 2io :: \odot EF + \odot AB : 2 \odot EF$ .

[Ib. cor. 10.] This cor. ought to be more exactly computed and demonstrated, because the following propositions depend thereon.

[Pr. 37, p. 109. And is therefore nearly equal, &c.] This resistance = weight of a cylinder (whose base is that little circle and altitude  $\frac{1}{2}IG$ , from which altitude the cylinder must fall, &c.) = force by which its motion may be generated, &c.

[Pr. 37, cor. 1.] Let  $F$  = force (mentioned in the prop.),  $M$  = medium's density,  $C$  = cylinder's density,  $V$  = velocity,  $D$  = diameter,  $T$  = time (of describing four times its length),  $R$  = resistance; then  $R = F \frac{\times M}{C}$  (by the prop.)

$$\propto C \times \frac{V}{T} \times D^2 \times \frac{M}{C} \propto V^2 D^2 M.$$

[Cor. 2.] Let EF be given, and density of the medium be also given. Then because this resistance = weight of a cylinder (whose base is PQ, and height  $\frac{1}{2}IG$ )  $\times \frac{EF^2}{EF^2 - \frac{1}{2}PQ^2}$ , therefore weight of this cylinder = resistance  $\times \frac{EF^2 - \frac{1}{2}PQ^2}{EF^2}$ . But the force requisite to take away the motion of this cy-

cylinder, whilst it moves four times its diameter  $\propto$  weight of this cylinder  $\times$  velocity  $\times$

$$\frac{1}{\text{time of describing four times its length}} \propto \text{resistance} \times \frac{\text{EF}^2 - \frac{1}{2}\text{PQ}^2}{\text{EF}^2} \times \frac{\text{EF}^2 - \text{PQ}^2}{\text{EF}^2} \times \frac{\text{EF}^2 - \text{PQ}^2}{\text{EF}^2}$$
. And the density of the cylinder being increased, the force is increased in the same ratio.

[Cor. 3] for since resistance : force ::  $\text{EF}^6 \cdot \propto$  density medium :  $\overline{\text{EF}^2 - \frac{1}{2}\text{PQ}^2} \times \overline{\text{EF}^2 - \text{PQ}^2}^2 \times$  density cylinder :: medium density :  $\frac{\overline{\text{EF}^2 - \frac{1}{2}\text{PQ}^2} \times \overline{\text{EF}^2 - \text{PQ}^2}^2}{\text{EF}^6} \times$  cylinder density : increase the distance it moves in any ratio, and the force (to destroy the motion) will be decreased in that ratio, as suppose the ratio of  $\frac{\overline{\text{EF}^2 - \frac{1}{2}\text{PQ}^2} \times \overline{\text{EF}^2 - \text{PQ}^2}^2}{\text{EF}^6}$ . Then re-

istance : force :: medium density : cylinder density.

[Schol. pr. 37, AE and BE (Fig. 43) be two parabolic arcs] since the resistance is the same, whether the fluid or the cylinder move, therefore the case of the resistance of the circle in Fig. 4 is the same case with the resistance of the circle AB in Fig. 6, considering the velocities HG and EG, where-with they are acted on ; but (by cor. 9, pr. 36) the resistance on PQ =  $\frac{1}{2}$  cylinder on PQ and height GH (through which a falling body acquires the velocity which the fluid has on PQ) ; but  $\frac{1}{2}$  this cylinder is = a parabolic spindle on PQ and height GH, nearly. Therefore the resistance on AB =  $\frac{1}{2}$  cylinder on AB and height GE (= GH), through which a falling body acquires the velocity with which the cylinder moves, which is equal to a parabolic spindle on AB and height GE = AEB, by construction [since *latus rectum* : (HG = ) GE :: GE : AG.] This is the very construction of Fig. 2.

[Ib. must be to this force as 2 to 3, at least,] for the weight upon the little circle (in cor. 7, p. 28), is there shewn to be equal to a cylinder of  $\frac{1}{3}$  the height GH at least. But the weight of a cylinder on that circle and  $\frac{1}{2}$  that height is the

force whereby the cylinder's motion may be generated in the time it moves four times its length, or  $2GH$ ; therefore the resistance in that cor. is to the weight here demonstrated (*cæteris paribus*), that is, to the force :: as  $\frac{1}{3}$  to  $\frac{1}{2}$  :: that is, as 2 to 3, at least; and increasing the density, the force is increased in the same ratio.

[Ib. let CF and DF.] It is necessary to allow as much time for the meeting of the particles of water (at the axis of the solid from the side thereof), after they are past the solid, as for their separation before they come to it. Now suppose this to be the case of Fig. 4. In the time that a particle passes from H to P, the same particle with the velocity in P will in the same time describe  $2HG$ , and therefore must pass twice as far to arrive to the axis again. Whence, in Fig. 6, the particle F must be twice as far from CD as E from AB; but the height of FCD = 2 height of AEB, by construction. For let  $z, y = 2$  ordinates to  $x = \frac{1}{2}AB$ ; and  $zz = 4rx = 4yy$ , and  $z = 2y$ .

[Cor. 1, pr. 38.] the demonstration is the same with that of cor. 1, pr. 37.

[Cor. 2, pr. 38.] Let density of globe : density of fluid ::  $d : 1$ ; then  $d \times \frac{4}{3}$  diameter of the globe = space fallen, and  $d \times \frac{2}{3}$  diameter globe = space it describes afterwards, in the time of its fall. Then, because forces (which generate a given motion) are reciprocally as the (times, that is, reciprocally as the) spaces which any body describes with the same uniform velocity, therefore the force of the globe's comparative weight (which generates its motion in the time it describes  $d \times \frac{8}{3}$  diameter) : force that generates the same motion (while it describes  $\frac{8}{3}$  diameter) ::  $\frac{8}{3}$  diameter :  $d \times \frac{8}{3}$  diameter :: 1 :  $d$  :: density of the fluid : density of the globe. Therefore, &c.

[Cor. 3, pr. 38.] The velocity being given, the motion M is also given; but (by this prop.) resistance  $\times d =$  force which will generate the motion M (in the time of moving  $\frac{8}{3}$  diameter); but this force must be = resistance, which will quite take away the said motion M (in the same time of moving  $\frac{8}{3}$  diameter). Now augment the time of moving (pre-

serving the velocity) in any ratio (let it = T); and this last resistance (which will quite take away the said motion M in the time T) will be decreased in the same ratio (= R). Therefore T and R are also given. Therefore by cor. 7, &c.

[Cor. 4, prop. 38.] for let  $M = \frac{2}{3}$  diameter in the time T; and because (by sup.)  $\frac{tM}{T+t} = \frac{1}{2}M$ , therefore  $T = t$ .

Now (by cor. 7, pr. 35) it will be  $(\frac{t}{T})^2 = 1 : (\log \frac{T+t}{T} \times 2)$ ,

$3, \&c. = \log. 2, \times 2, 3, \&c. = ,693 : : \frac{2}{3}$  diameter :  $\frac{5.545}{3}$

diameters, which is less than two diameters. Now that M ought to be  $= \frac{2}{3}$  diameter in the time T is plain; for it is required (by cor. 7, pr. 35) that the resistance R quite takes away M in the time T. But (by this pr. 38) the resistance (= force required) will just take away the motion M in moving  $\frac{2}{3}$  diameter. But if T be any other time than that of moving  $\frac{2}{3}$  diameter, then the force to destroy M in that time will not be = resistance, but greater or lesser than it.

[Pr. 40, let the globe, &c.] This is calculated in prob. 14 of my book of Fluxions. In cor. 2 of that prob. G, H, and

N, are found, and thence the velocity  $= \frac{N-1}{N+1} H$ , where H

is the greatest velocity the globe can acquire in falling in the fluid; and the height fallen comes out the same in value with his, but in other symbols; but the one may be easily reduced to the other.

[lb. the numbers in the fourth column, &c.]  $2F$  is the space described in the time  $G$  with the greatest velocity  $H$

(by the prob.); therefore  $G : 2F :: P : \frac{2P}{G} F =$  the space described in the time  $P$  with the greatest velocity, the numbers in the fourth column.

[Sch. pr. 40, and any other globe, &c.] that is, the magnitude of any body  $\propto$  the excess of its weight in vacuo above that in water. Or  $132,8 : 1 ::$  this said excess : magnitude of the globe. Hence diameter globe (in Exp. 1.)  $= 0,84224$ .

[Ib. exp. 1, but this space, by reason, &c.] to demonstrate this, it is necessary to premise, that the greatest velocity wherewith a globe can descend by its comparative weight through a fluid in a canal is that which it may acquire by falling with the same weight, without resistance, and in its fall describing a space which is to  $\frac{4}{3}$  diameter as  $mn^2d$  to  $l^3$ . Putting density globe : density fluid : :  $d$  : 1, and  $l$  = orifice of the vessel,  $N = l$  — great circle of the globe,  $m = 1 - \frac{l}{2}$  great circle. For (as in cor. 2, pr. 38) the space it afterwards describes (in the same time of its fall)  $= \frac{mn^2d}{l^3} \times \frac{8}{3}$  diameter; then the force of the globe's comparative weight (which generates its motion in the time it describes  $\frac{mn^2d}{l^3} \times \frac{8}{3}$  diam.) : force (that generates the same motion in the time it describes  $\frac{8}{3}$  diameter) :: ( $\frac{8}{3}$  diameter :  $\frac{mn^2d}{l^3} \times \frac{8}{3}$  diameter ::)  $l^3 : mn^2d$ ; therefore (by pr. 39) the resistance of the globe = force of the globe's comparative weight (which generates this motion in the time it describes  $\frac{mn^2d}{l^3} \times \frac{8}{3}$  diameter, or falls  $\frac{mn^2d}{l^3} \times \frac{4}{3}$  diameter); and, therefore, this force cannot accelerate the globe. Q.E.D. Now,

By the experiment,  $2F = d \times \frac{8}{3}$  diameter; but the space the globe uniformly describes (with the greatest velocity wherewith it can descend in the canal or vessel, and in the time of acquiring it with its compound weight)  $= \frac{mn^2d}{l^3} \times \frac{8}{3}$  diameter  $= \frac{mn^2}{l^3} \times 2F = \frac{mn^2}{l^3} \times 4.4251$ . Also  $\sqrt{95,219} :$   
 $\sqrt{\frac{mn^2}{l^3}} \times F :: 1'' : \sqrt{\frac{mn^2}{l^3}} \times \sqrt{\frac{F}{95,219}} = \sqrt{\frac{mn^2}{l^3}} \times G =$   
 $\sqrt{\frac{mn^2}{l^3}} \times 0,15244$  (by the laws of falling bodies) = time of acquiring the greatest velocity. And therefore  $(\sqrt{\frac{mn^2}{l^3}} \times G$

(= time) :  $\frac{mn^2}{l^3} \times 2F$  (= space) :: ) G :  $\sqrt{\frac{mn^2}{l^3}} \times 2F$  ::  
 $4'' : \frac{n}{l} \sqrt{\frac{m}{l}} \times \frac{8F}{G} = (\text{space described in } 4'') = \frac{n}{l} \sqrt{\frac{m}{l}} \times$   
 $116,1245$ . Subduct  $\frac{mn^2}{l^3} \times 1.3862944F = \frac{mn^2}{l^3} \times 3,0676$  ;  
and  $\frac{n}{l} \sqrt{\frac{m}{l}} \times 116,1245 - \frac{n}{l} \sqrt{\frac{m}{l}} \times 3,0676 = \text{space fallen}$   
in the fluid,  $= 116,1245 - 3,0676 \times \frac{n}{l} \sqrt{\frac{m}{l}}$ , nearly, as  
the author makes it.

[Ib. ex. 13.] This and the following are upon the same computation as the foregoing experiments in water; to wit, the theory of non-elastic fluids: the reason is, because he considers fluids as elastic whose parts are discontinued, &c. See schol. pr. 35.

[Ib. p. 128, equal to  $\frac{1}{4586}$ , &c.] This is plain from what follows afterwards; for let a space :  $\frac{8}{3}D$  :: 860 : 1; this space  $= \frac{6880}{3} D = 2293D$ , and this space the globe moves in the time T with velocity V; let, therefore,  $T = 2293$ , and  $t = \frac{1}{2}$ . Then motion lost  $= \frac{\frac{1}{2}V}{2293 \times \frac{1}{2}} = \frac{1}{4587} V$ . Note;  $t$  ( $= \frac{1}{2}$ ) is not sensibly increased by the resistance in moving through  $\frac{1}{2}D$ ; and hence I have taken it for the same.

[Ib. let D be the, &c.] for, put density globe : density fluid ::  $d : 1$ ; also let the space  $\frac{8d}{3} D = S$ . Now, by prop. 38,  $d \times$  resistance = force, which (uniformly continued) will quite destroy its motion in the time it moves  $\frac{8}{3}D$ ; therefore  $d \times$  resistance will quite take away its motion whilst it moves  $\frac{8}{3}D$ . But (because the force of that resistance and the time of its acting are reciprocally proportional) the resistance the globe meets with will quite take away its motion whilst it moves (with V)  $\frac{8d}{3} D = S$ . Therefore (by cor. 7, pr. 35) the globe

in the time  $T$  (of moving  $S$ ) meets with a resistance  $R$  which will quite destroy its motion  $V$ ; therefore in the time  $t$  it will lose  $\frac{t}{T+t} V$ , &c.

### SECTION VIII.

[Pr. 42, and (Fig. 44) because the motion of the waves, &c.] Let the waves A, B, C, move from A towards C, and suppose the water to descend from the vertex C into the hollow E, and thence up to B, which it will do, supposing the vertices A, B, C, to be of equal altitude; also let the water descend from B to D, and ascend to A, &c. Now since the water in the vertex B descends towards D at the same time that other water (that had descended from C to E) is ascending to B, therefore the water will be lower at the point B and higher at the point b than before; and consequently that the vertex of the wave B will be transferred towards C. Therefore the vertices (and hollows, and any corresponding points) of all the waves will move from thence towards C; therefore the waves will be carried on by the deflux of water down CE, BD; and the accumulation thereof on EB, DA.

[Pr. 43, case 1, at nearly equal distances, &c.] they set forward at equal distances, and continue so, because they move with equal velocity, as appears by (the demonst. of case 1, 2, 3, of prop. 48. (Fig. 45.)

[Pr. 46, because the motion of the waves is carried on, &c.] Let A be the vertex of a wave. The fluid at b is more pressed (by reason of a greater depth of incumbent water) than d; therefore that greater pressure causes the water to recede from b in all directions, and to ascend at c; therefore the vertex of the wave is now in c. Also a greater pressure of water propagated from d to f causes the water to descend at c, and ascend at e; therefore the vertex of the wave moves along a, c, e, &c. And this it will always continue to do, if at first any force is supposed to be applied to the side gha, to hinder the ascent of the water there in like manner as it ascends at c, e; for then the side gha will be afterwards always descending, and the side ace ascending, without the continuance of such force,

[Ib. transverse measure] is measuring along the surface of the water; as is plain from the demonstration.

[Ib. cor. 2, rather performed in a circle] as in the note on prop. 42 (Fig. 46). But if the breadth of the waves be measured not in the right line acb, but along the surface of the water (because the deflux of water through b, c, a, is made in a curve line near a circle), then the time mentioned in the prop. will be very nearly as it is there assigned.

[Pr. 47] Fig. 1, Pl. 9, is wrong. It should be thus (Fig. 47); [and the original is wrong, too] where l, m, n, stand at N, M, L.

[Ib. — with those of an oscillating pendulum.] for, by prop. 52, lib. 1, space described by a pendulum in the time PI is to a whole vibration as the time PI to the time PIS; and to two vibrations as PI to PHSP; and the accelerating force of a pendulum is as its distance from the lowest point. See cor. pr. 51.

[Ib. that is, (because HL — KN is,) &c.] that is, HL — KN : HK :: OM : OI.

[Pr. 48, case 1, distances of the pulses, &c.] the distances of the pulses in one medium be = distances of pulses in the other.

[Ib. and therefore the pulses;] hence it appears that the correspondent parts in both mediums (vibrate or) go and return together, or in equal times, though one makes longer vibrations than the other. But (by supposition) the distances of pulses in both mediums are equal; and the pulses in both are translated through these equal distances in the times of the parts of the mediums (vibrating or) going and returning, which times are proved equal; therefore the pulses in both mediums move equal distances in equal times, and therefore are equally swift.

[Pr. 48, case 2, then will their contractions, &c.] be equal, in equal spaces.

[Ib. and moreover, &c.] for space  $\propto \frac{\text{square of the time}}{\text{matter}}$   
(for these spaces are generated by accelerative forces), and

time  $\propto \sqrt{\text{space}} \times \sqrt{\text{matter}} \propto \sqrt{\text{space}} \times \text{space} \propto \text{space}$ .

[Ib. case 3, the time which is necessary, &c.] for space  $\propto \frac{\text{time} \times \text{force}}{\text{matter}} \propto \frac{\text{time} \times \text{force}}{\text{density}}$ . Therefore if the space

be given, time  $\propto \sqrt{\frac{\text{density}}{\text{force}}}$ . Therefore velocity  $\propto \frac{1}{\text{time}}$   
 $\propto \sqrt{\frac{\text{force}}{\text{density}}}$ .

[Pr. 49. — ratio of PO to A conjunctly;] that is, time  $\sqrt{\frac{\text{matter}}{\text{force}}} \propto \frac{1}{\sqrt{\text{force}}}$ , when the quantity of matter is given (this in accelerated motions). Then time of one vibration by that elastic force : time of one vibration of a pendulum PO, by the weight ::  $\sqrt{VV} : \sqrt{PO \times A}$ ; and time of one vibration of PO, by the weight : time of one vibration of A ::  $\sqrt{PO} : \sqrt{A}$ . Therefore, &c.

[Ib. — of the going and returning] of the pendulum PO.

[Ib. — of one oscillation composed of the going and returning,] of the pendulum A.

[Ib. cor. 1, — to its circumference.] For (Mechanics, prop. 24) diameter : circumference :: time of fall through  $\frac{1}{2}$  radius : time of one vibration. Therefore as radius : circumference :: time of fall through  $\frac{1}{2}$  radius : time of two vibrations.

[Ib. cor. 2.] for velocity  $\propto \frac{\text{space}}{\text{time}} \propto \frac{A}{\sqrt{A}} \propto \sqrt{A} \propto$

$\sqrt{\frac{\text{e. force}}{\text{density}}}$ . By elastic force he means that which arises by heat, as well as that arising from the compression.

[Pr. 50. Let the number] of the double vibrations, &c.

[Ib. schol. in the subduplicate ratio of the defect of the matter] that is, of the quantity of vapour or watery particles. And (by prop. 49, cor. 2) the velocity will be increased as (the  $\sqrt{\text{density}}$ , that is, as) the  $\sqrt{\text{quantity of true air in a given space}}$  is decreased,

[Ib. near 100 pulses in] This computation pre-supposes that those vibrations of the string are double vibrations, or vibrations composed of its going and return.

### SECTION IX.

[Pr. 51, but the differences of the angular motions, &c.] these are as the difference of the absolute motions round the axis; that is, as the difference of the absolute translations, or as the relative translations (which are those here spoken of) directly, and the distances inversely.

[Ib. of quadratures of curves, &c.] Let  $SD = x$ ;  $Dd = y$ ; then  $Dd \propto \frac{1}{xx} \left( = \frac{b}{xx} \right)$ , by construction; therefore  $y = \frac{b}{xx}$ ; and  $yx = bx^{-2}x$ ; and  $\frac{bx^{-1}}{-1} = -\frac{b}{x} = \text{area}$ ; therefore area  $\propto \frac{1}{x} = DdQ$ .

(Ib. cor. 3.) In a given time let the parts 1, 2, 3, of the fluid describe the (Fig. 48) spaces 1r, 2s, 3t; take away or add the equal angular motions 1n, 2o, 3p; so that the spaces nr, os, pt, be described in that given time: now whether the points 1, 2, describe the spaces 1r, 2s, or nr, os, the translations at the end of the given time will be  $= si$ ; also if 2, 3, describe 2s, 3t, or os, pt, the translations at the end of the given time will be the same in either case, viz.  $= ut$ ; therefore the translations being equal, and the attrition also, the motions will be continued.

[Pr. 52, cor. 8, about any given axis] passing through the globe; the relative motion of the parts of the globe and fluid are the same still: therefore, &c.

[Ib. cor. 9, semi-diameter of the globe;] let BC be a plane (Fig. 49), D a point of the vessel, A a point of the globe; then period. time of A : time B :: < velocity B : < velocity A; and therefore time A + time B : time A :: (velocity B + velocity A =) < velocity A from B : < velocity B :: time B : time A from the plane BC (to BC again) ::  $CB^2 : CA^2$ . But time B from DC = time D from BC; therefore time D (from BC) : time A (from BC) ::  $CB^2 : CA^2$ ; as it should be by the prop. and cor. 8.

## BOOK III.

[Pr. 3, the action of the sun, attracting, &c.] see cor. 7, pr. 66, lib. 1, or cor. 14, where  $KL \propto PT$ , nearly (SK or ST being given); and the force TM in its mean quantity is  $= PT$ , nearly.

[Pr. 4, and the space which a heavy body describes] by falling in 1": half the length of the pendulum ( $= \frac{1}{2}$  radius) :: square of 1": square of the time of falling through half the pendulum :: (Mechanics, pr. 24) square of the circumference : square of the diameter.

[Ib. p. 169, line 17, the mean distance of 60 diameters] it should be 60 semi-diameters, as it is in the original.

[Pr. 6, p. 174, — subduplicate of that proportion, as by some computations I have found,] i. e. as I found by making some calculation.

For let R be the distance of the Sun and Jupiter, d the greater force, e the lesser; then, to find the distance x where d may be diminished to e: Because the forces are reciprocally as the squares of the distances, say  $\frac{1}{R^2} : \frac{1}{x^2} :: d : e :: x^2 : R^2$ ; and  $x : R :: \sqrt{d} : \sqrt{e}$ ; therefore at the distance x the weight d (of the satellite) would become equal to e; and the centre of their orbits would not be in Jupiter, contrary to experience. And if the force of the satellite was less, by a like computation, the centre of their orbits would be nearer than Jupiter, also contrary to experience.

[Pr. 8, cor. 1, and  $\frac{1}{169282}$  respectively] this should be  $\frac{1}{196282}$ ,

or rather  $\frac{1}{194075}$ .

[Ib. at the distances 10000, &c.] these are as the diameters; and the apparent diameters are found by astronomical observation, which,  $\times$  by the proportioned distances, gives the ratio of the real diameters.

[Pr. 8, cor. 4, — of so much the greater density.] This is not generally true, as appears by cor. 2 and 3.

[Ib. cor. 3, — truly defined.] This is demonstrated in Gravestrand, l. IV. p. 232-3. See Appendix, p. 20.

[Pr. 10, lose almost a tenth part of its motion.] For, by schol. pr. 40, p. 128, it is,  $1 : 860 :: \frac{g}{3} : 2293.33 \dagger =$  space described in the time  $T$ ; putting  $D = 1$ ,  $V = 1$ ; and  $T : t :: 2293.33 : 2293.33 \frac{t}{T} =$  space (uniformly) described in the time  $t$ . Then (p. 129)  $\frac{t}{T} : 2,302, \&c. \times \log. \frac{T+t}{T} :: 2293.33 \frac{t}{T} : 229.5$ ; therefore  $2293.33 \times 2.30258 \times \log. \frac{T+t}{T} = 229.5$ ; and  $\log. \frac{T+t}{T} = ,043461$ ; and therefore  $\frac{T+t}{T} = 1.105\dagger$ ; and  $\frac{T}{T+t} = \frac{1}{1.105} = \frac{9}{10} \dagger$ , and  $\frac{t}{T+t} = \frac{1}{10}$ , nearly.

[Ib. or as  $75(0^{12})$  to 1 nearly.] In Fig. 3, Pl. 5, l. II. Let  $SA =$  rad. earth  $= 20949674$  feet,  $AB = 1$ ,  $AF = 200$  miles  $= 1056000$  feet; then, by cor. pr. 22, II.  $Aa - Bb : Aa - Ff :: \text{thin} : \text{thnz}$ ; that is,  $AB \times SF : AF \times SB ::$ , or  $1 : 1005325 :: \text{thin} : \text{thnz} ::$  (by sch. pr. 86, hyperbola)  $\log. AH - \log. BI : \log. AH - \log. FN$ ; because water is 860 times denser than air, and the incumbent weight of the atmosphere at the surface of the earth is about  $=$  weight of 33 feet of water; therefore the weights (and consequently the densities) in A and B are as 28380 and 28379; whence  $AH$  or  $St = 28380$ , and  $BI$  or  $Sn = 28379$ ; and  $\log. AH - \log. BI = ,0000152$ ,  $\log. AH = 4,4530124$ ; therefore  $1 : 1005325 :: ,0000152 : 4,4530124 - \log. FN$ ; and  $\log. FN = -11,1730$ . And therefore  $FN = ,000000000015$ , and density in F : density in A ::  $15 : 2838000000000000 ::$ , or as 1 : to  $1800000000000000$ , which is less than 1 to 75000000000000; but the computation will vary as the radius of the earth, the density of water and air at the earth, is supposed to vary.

[Ib. and hence the, &c.] as before (by pr. 40, schol.), since density of water : to density of air 200 miles from the earth ::  $645(0^{14}) : 1$ . Space described in time  $T = 172(0^{15})$ . And

$\frac{t}{T} : \log. \frac{T+t}{T} \times 2.302$ , &c. :: 172 (0<sup>15</sup>)  $\times \frac{t}{T}$  (= space described in time t) : 2792250000 = space Jupiter describes in 1000000 years. Therefore  $\log. \frac{T+t}{T} = ,0000000705$ . And  $\frac{T+t}{T} = 1.000000017$ ; and  $\frac{T}{T+t} = ,999999986$ ; and  $\frac{t}{T+t} = ,000000014$ , which is less than  $\frac{1}{1000000}$ .

[Pr. 13, are almost as 16, &c.] for (by this and cor. 2, pr. 8) accelerative force of Jupiter towards Saturn : accelerative force of Jupiter towards the Sun ::  $25 \times \frac{1}{3021} : 16 \times 1 :: 81 \times \frac{16 \times 81 \times 3021}{25}$ .

[Pr. 14, schol. And hence we may find, &c.] For, by cor. 7, pr. 66, I. the apsides of their orbits move in consequentia. And, by cor. 16, of the same, the motion of the apsides of the body P (Pl. 21, Fig. 2) will be as the periodical time of P directly, and the square of the periodical time of T inversely; that is, as the periodical time of P (because the periodic time of T round S, or of the Sun round Jupiter or Saturn, or (which is the same thing) of Jupiter or Saturn round the Sun, is given); that is (by prop. 15, I.), in the sesquiplicate ratio of their distances PT.

[Pr. 19, in the duplicate proportion of the rad. to the cosine of lat.] (Fig. 50) for (by cor. 3, pr. 4, l. I.) centrifugal force at the equator : centrifugal force at a from the axis ( $= ab$ ) :: rad. : cos. lat. :: r : c. Also the force directly from the axis : force directly from the earth :: ab : be :: r : c. Therefore, *ex equo*, centrifugal force at the equator : centrifugal force directly from the earth, at a :: rr : cc.

[Ib. but by computation (from] Let (Fig. 51) GC = t = 101, BP = C = 100, PE = x. Then will  $ED^2 = \frac{tt}{cc} \times cx - xx$ . And  $ER^2 = PD^2 = \frac{tx}{c} - \frac{tt}{cc} x^2 + x^2$ . And

$$ER = \sqrt{\frac{tt}{c} - \frac{cc - tt}{cc} x^2} = (\text{suppose to}) \sqrt{r^2x - sx^2}.$$

Whence the fluxion of the area PQRE is  $= \overline{r^2 x - 8x^2} \frac{1}{2} x = rx \frac{1}{2} x - \frac{8x^3}{2r} x - \frac{s^2 x \frac{5}{2}}{2.4r^3} x - \frac{3s^3 x \frac{7}{2}}{2.4.6r^5} x - \frac{3.5s^4 x \frac{9}{2}}{2.4.6.8r^7} x - \text{ &c.}$

And the area PQRF  $= \frac{2}{3}rx \frac{3}{2} - \frac{sx \frac{5}{2}}{5r} - \frac{s^2 x \frac{7}{2}}{7.4r^3} - \frac{3s^3 x \frac{9}{2}}{9.4.6r^5} -$

$\frac{3.5.s^4 x \frac{11}{2}}{11.4.6.8r^7} - \text{ &c.} = \frac{2}{3}rx \frac{3}{2} - \frac{3.sx}{2.5r^2} A + \frac{5.sx}{7.4r^2} B + \frac{7.3.sx}{9.6r^2} C$

$+ \frac{9.5.sx}{11.8.r^2} D + \text{ &c.} = (\text{putting } \frac{sx}{rr} = q) \frac{2}{3}rx \frac{3}{2} - \frac{3}{10} Aq + \frac{5}{28} Bq + \frac{7}{18} Cq + \frac{45}{88} Dq + \text{ &c.} = (\text{when } x = c = 100; \text{ and } s \text{ being } = \frac{cc - tt}{cc} = ,0201; \text{ and } r^2 = \frac{tt}{c} = 102,01. \Rightarrow) \text{ to}$

6693,39. And PBM  $= 5000$ . Whence PQRM  $= 1693,39$ .

Therefore, by cor. 2, pr. 91, I. force of the spheroid : force of the sphere :: (when P and A coincide) AS  $- \frac{AS \times PQRM}{CS^2}$

$$\therefore \frac{AS}{3} :: 1 - \frac{PQRM}{CS^2} : \frac{1}{3} :: 1 - \frac{4 \times 1693.39}{10201} : \frac{1}{3} :: 1,003.$$

$$\therefore 1 :: 126 : 125.$$

[Ib. p. 190, if the density, &c.] this will appear by considering what went before, and cor. 3, pr. 91, I.

[Ib. but if the diurnal, &c.] for the (centripetal or) centrifugal force is as the square of the velocity when the rad. is given.

[Ib. 191, — augmented in proportion as the force of gravity is diminished] for, reviewing the former calculation, if the density of the earth were greater than it is, the force of gravity to the centrifugal force would be greater than in that proportion of 289 to 1 (and the two diameters would be in a less proportion than 289 to 288; and also the diameters would be in a less proportion than 230 to 229); consequently the ratio of the difference of the diameters to either decreases as the gravity increases; and, on the contrary, that ratio increases as the gravity decreases. And upon this depends the

foregoing proportion, as  $\frac{4}{505} : \frac{1}{100} :: \frac{1}{289} : \frac{1}{229}$ .

[Ib. line 15] read  $\frac{29}{5} \times \frac{400}{94\frac{1}{2}} \times \frac{1}{229}$  to 1. In the original, 2d edit. it is  $\frac{29}{5} \times \frac{5}{1} \times \frac{1}{229}$  to 1, or as 1 to 8.

[Pr. 20 (Fig. 52). Whence arises this theorem] for let nkyn be a circle circumscribed round the ellipsis nhyl; then  $khm : fbc (=) abd :: kt^2 : fg^2$ . And  $ab = \frac{khm \times fg^2}{kt^2 \times bd}$ . But, because  $khm$  and  $kt$  are given, and  $bd =$  (nearly to)  $2kt$ , which is given, therefore  $ab \propto fg^2$ : but the difference between the weight at n and a is as (at — bt =) ab. Therefore the increase of weight is as  $fg^2$ , or the square of the fine of lat. at  $\alpha$ , nearly, or (by trigonometry) as the verf. of  $2an$ .

[Ib. and the arcs of the degrees, &c.] The length of a degree is  $\propto$  rad. of curvature, and in the points n and b are (by ex. 1, pr. 19, sect. II. curve lines) as the parameters of ny and hl; that is, as  $hl^3$  to  $ny^3$ , or  $229^3$  to  $230^3$ ; that is, as 56637 to 57382, as is inserted in the table, p. 247.

But to find the rad. curvature at b, we have (by ex. 2, pr. 5, sect. II. Fluxions)  $mH = y + \frac{4 \times a - b}{abb} y^3$ ; and, therefore (putting s for the fine of the angle nt<sub>b</sub>), rad. curvature  $mC$ , at  $b = \frac{y}{s} + \frac{4 \times a - b}{abbs} y^3 =$  (because  $\frac{y}{s}$  is a given ratio)  $R + \frac{4 \times a - b}{abb} Ry^3$ , very near, R being the radius of curvature in n; and, therefore, the increase of this radius, which is as the increase of the degrees, is as  $yy$  or  $ss$  nearly. And here  $a - b$  being very small,  $\frac{4 \times a - b}{abb} R$  is an extremely small quantity.

[Pr. 23, — which I cannot here descend to explain.] Since the Moon's orbit is more eccentric than those of Jupiter's satellites, the motion of the Moon's apogee will be greater in proportion than the motion of the apogee in any satellite.

[Pr. 27. The area, &c.] (Fig. 53) For the moment of the area is  $\propto cb \times ad$ . But  $ad \propto$  angle  $bcd \times cb =$  ho-

rary motion  $\times$  cb. Therefore area  $\propto$  cb<sup>2</sup>  $\times$  horary motion.

[Pr. 28. But the attraction of the moon ;] for let A = attraction of P towards T. Let the attraction or force LS be resolved into the two, LM, SM, the first acting towards M, the latter towards S. Then the whole attraction of P towards T is A — ST + LM — TM. Let A — ST = F. Then the attraction of P towards T in the syzygies is F — 2AT; and in the quadratures F + CT. By prop. 25, F : ML :: 178725 : 1000. And the forces 2AT and CT will be to each other as 2000AT and 1000CT; that is, as

$\frac{2000}{CT \times N}$ , and  $\frac{1000}{AT \times N}$ . And the force F in the syzygies and quadratures will be as 178725 directly, and the square of the distance from T reciprocally. Therefore, &c. (Note, +  $\frac{2000}{CT \times N}$  is falsely printed for  $-\frac{2000}{CT \times N}$ , in line 11, p. 210.)

[Ib. — Quadrature in C; or, which —] for CTP : CTp :: angular motion of the moon from the sun's quadrature : to its angular motion from the fixed point C; that is (because the given circumference, or one revolution, is to be described in either case), :: periodic revolution : synodic revolution.

[Ib. But by computation I find,] with (Fig. 54) the radii TA, TC, describe the circles AIC, Cie; and let AZ, CZ, touch the ellipsis in A and C; and take AI to Ac, and Ci to Cc, as  $\angle$  CTP to CTp (Fig. 5, Pl. 10). Now in order to determine the curvature of the orbit, suppose the point a to coincide with A, and draw the parts of the orbit ARO, Cro, which will pass between the ellipsis and circle; for TO or TE is less than TL, and greater than (TA or) Te; and To (that is, Te) is less than TC or Te, and greater than TI. Draw the lines TB, Tb, so that AB = cb; and let the points A, I, C, and also C, i, e, coincide; and the curvatures of the ellipsis, orbit, and circles, at the points A, C, will be respectively as BE, BR, BI, and be, br, bi. And the difference of curvatures of the ellipsis and circles, and of the orbit and circles in A, C, will be as EI, RI, and ei, ri, respectively. But RI : EI or OC (because TE = TO, and TI = Te ::) AI<sup>2</sup> : Ac<sup>2</sup> (lem. 11, I.)

:: CTP<sup>2</sup> : CTp<sup>2</sup>. Whence RI = EI  $\times$   $\frac{CTP^2}{CTp^2} = \frac{BI - BE}{BI + BE}$

$\times \frac{CTP^2}{CTp^2}$ . Also  $ri : oc$ , or  $ei$ , or  $be - bi :: Ci^2 : Cc^2 :: CTP^2 : CTp^2$ .  $CTp^2$ . Whence  $ri = \overline{be - bi} \times \frac{CTP^2}{CTp^2}$ . But in the  $\odot$  TA, BI, AI, or AB, and  $2AT$ , are  $\ddots$ ; and in the  $\odot$  TC,  $bi$ ,  $Ci$  or  $Cb$ ,  $2CT$ , are  $\ddots$ ; whence  $BI = \frac{AB^2}{2AT}$ , and  $bi = \frac{Cb^2}{2CT}$ . Also in the ellipsis BE  $\times 2AT : BA^2 :: AT^2 : TC^2$ ; and  $be \times 2CT : bC^2 :: CT^2 : AT^2$ . Whence  $BE = \frac{BA^2 \times AT}{2CT^2}$ ; and  $be = \frac{bC^2 \times CT}{2AT^2}$ . Now from hence the curvature of the orbit at a : curvature of the orbit at C :: BR, or  $BI - IR : br$  or  $bi + ir :: \frac{AB^2}{2AT} - \overline{BI - BE} \times \frac{CTP^2}{CTp^2} : \frac{Cb^2}{2CT^2} + \overline{be - bi} \times \frac{CTP^2}{CTp^2} :: \frac{AB^2}{2AT} \times \frac{CTp^2}{CTP^2} - BI + BE : \frac{Cb^2}{2CT^2} \times \frac{CTp^2}{CTP^2} + be - bi :: \frac{AB^2}{2AT} \times \frac{CTp^2}{CTP^2} - \frac{AB^2}{2AT} + \frac{BA^2 \times AT}{2TC^2} : \frac{Cb^2}{2CT^2} \times \frac{CTp^2}{CTP^2} + \frac{bC^2 \times CT}{2AT^2} - \frac{Cb^2}{2CT} :: \frac{CTp^2}{AT \times CTP^2} - \frac{1}{AT} + \frac{AT}{TC^2} : \frac{CTp^2}{CTP^2 \times CT^2} + \frac{CT}{AT^2} - \frac{1}{CT} :: \frac{CTp^2 - CTP^2}{AT \times CTP^2} + \frac{AT}{TC^2} : \frac{CTp^2 - CTP^2}{CT \times CTP^2} + \frac{CT}{AT^2} :: AT^3 + \frac{CTp^2 - CTP^2}{CTP^2} \times AT \times CT^2 : CT^3 + \frac{CTp^2 - CTP^2}{CTP^2} \times CT \times AT^2$ .

[Pr. 29. The tangent of the angle (Fig. 55) CTP, &c.] On the axis DC describe the circle CdD, and let the line Tb revolve uniformly to d in the same time that TP revolves to A, and describes areas proportional to the times in the ellipsis CAD. Draw be  $\parallel$  to  $dT$ , and draw Tb, TP. Since the circle and ellipsis are described in equal times, parts proportional to the whole will be described in any equal times. Now area ellipsis : area circle :: TA : Td ( $= TC$ ) :: eP : eb :: (ePC + ePT  $=$ ) TPC : (ebC + ebT  $=$ ) TbC; therefore the point in the circle is at b when the point in the ellipsis is at

P. But  $TA : (Td) = TC :: eP : eb :: \text{tangent} < eTP : \text{tangent} < eTb$  ( $= <$  of the mean motion).

[Ib. Square of the sine of the angle CTP.] These things are plain from prop. 26, where  $Pd$  (Fig. 4) is the excess of the moment, and is as  $\frac{PK^2}{PT}$ .

[Ib. which we may effect, &c.] (Fig. 56) Let  $A$  be the area described by the moon in any time, arc  $CP = z$ ,  $r = \text{radius } TC$ ,  $PK = y$ ,  $TK = u$ ,  $t = \tan. PTC$ . Then  $A = \frac{yyz}{r} = yx$ , and  $A = \text{Fl. : } yx = \text{area } CPK$ . But  $TPC$  expresses the mean motion, and therefore  $TPK$  is the equation, which is as  $TK \times PK$ , or  $uy$ .

But, by the nature of the circle,  $\dot{z} = \frac{rrt}{rr + tt}$ ; and since  $t$  is in a given ratio to  $t$  (or as .3123 to 69), therefore  $\dot{z}$  is as  $\frac{rrt}{rr + tt}$  or  $\frac{t}{\sqrt{rr + tt}} \times TK$ , or  $y \times TK$ , or as  $uy$ . Therefore, decreasing the tangent in the subduplicate ratio of 11073 to 10973, or in the simple ratio of 69 to 68, 6877; accelerates the area in proportion of  $PK^2$ , as it ought to do.

[Ib.— in a proportion compounded of the duplicate, &c.] for (by cor. 16, pr. 66, l. I.), all angular errors are as the square of the time of the moon's revolution directly, and the square of the time of the earth's revolution inversely; that is (by pr. 15, l. I.), as the square of the time of the moon's revolution directly, and the cube of the earth's distance from the sun inversely.

[Pr. 30. And this force, by prop. 25, is, &c.] The force  $3PK : \text{force } ML :: 3IT : PT$  (for these are the same), and force  $ML$ : centripetal force by which the moon revolves, &c. :: 1 :  $178\frac{2}{3}$ . (by pr. 25); therefore, *ex equo*, force  $3PK$ : centripetal force the moon revolves with ::  $3IT \times 1 : PT \times 178\frac{2}{3}$ ; or as  $IT : \text{rad.} \times 59,575$ .

[Ib. the half of which the moon] it should be, which the moon, by the action of the said force, as it is in the first edition.

[Ib. And the angle PTM (Fig. 57) is equal to the angle, &c.] for in this case  $LM$  is perpendicular to  $MP$ . Let  $PR$

be a tangent to the point P; then the triangle RPT is a right one; and angle RPM = angle PTM. And angle LPM :  $\angle(RPM =) \angle PTM$  (when the radius is PM) :: LM : RM :: force producing LM : force producing RM :: 1 : 59,575.

[Pr. 31, cor. and the decrement is to the remaining motion as 100, &c.] The motion of the nodes in the octants : motion in the syzygies :: 11073<sup>2</sup> : 11023<sup>2</sup> :: (because 11073, 11023, 10973, are in arithmetical, and nearly in  $\frac{1}{2}$  progression) 11073 : 10973; and decrement : remaining motion :: (11073 - 10973 =) 100 : 10973.

[Ib. but the decrement] (Fig. 58) for the decrements are as the forces and times, or as the whole motions and times; that is, decrement at H (or increment at h) : decrement at A :: mot. at H, and time : mot. at A, and time :: (that is, by what went before) as mot. at H  $\times \sqrt{yy - \frac{1}{4}rr}$  : mot. at A  $\times rr - \frac{1}{2}rr$  (or mot. at A  $\times \frac{1}{2}rr$ ). This reasoning being very obscure, you will find the motion of the nodes clearly investigated in (prop. 5, sect. 6, and cors. of) my Astronomy.

[Pr. 32, it is drawn back again] that is, supposing the place of the node given.

[Pr. 32. Now the area of the semi-circle] For let NT = 1, AZ = y, TZ = x. Then  $eZ = \frac{y^3}{9.0827 + yy}$ , and Ae (AZ - eZ) =  $\frac{9.0827y}{10.0827 + yy}$ . But  $yy = 1 - xx$ , therefore  $Ae = \frac{9.0827\sqrt{1 - xx}}{10.0827 - xx} = \frac{b-1\sqrt{1 - xx}}{b - xx} = \frac{b-1}{b} \times \frac{1}{b} + \frac{xx}{bb} + \frac{x^4}{b^3}$ , &c.  $\times$  into  $\sqrt{1 - xx}$ , putting  $b = 10.08276$ , and the fluxion area  $NAe = \frac{b-1}{b} \times \frac{1}{b} + \frac{xx}{bb} + \frac{x^4}{b^3}$ , &c.  $\times \dot{x}$   $\sqrt{1 - xx}$ ; whose fluent (by form 17, Fluxions) =  $\frac{1}{b} + \frac{1}{4bb} + \frac{1}{8b^3}$ , &c.  $\times \overline{b-1}$ .  $\phi$  (putting  $\phi =$  quadrant, or  $\frac{NAn}{2}$ ) = .92435  $\phi$ , therefore half the area  $NFn = .07565\phi$ . And the area  $\phi$  (or quadrant) is to the area  $\frac{NFn}{2}$ , or the semi-circle to the whole area  $NFn$ , as 1 to .07565, or as 793 to 60.

[Prop. 33, will nearly agree with] let  $N = 19\ 49\ 3\ 55$ ,  
 $p =$  periphery of  $NA$  or  $DFB$ ,  $E$  the equation,  $S = S.2NA$ .

Then  $NAZ$  represents the mean motion of the nodes for the arc  $NA$ , as  $N$  does for the whole circle, and  $NAZ$  is the true motion; and  $ATZ$  is the difference proportional to the equation. Divide by  $\frac{1}{2}r$ , and then  $p : \frac{xy}{r} :: N : E$ , or  $p : \frac{2xy}{r}$   
 $:: \frac{1}{2}N : E$ , that is,  $p : S :: \frac{1}{2}N : E$ ; whence if  $BF = 2NA$ , then (by construction)  $AD : CD :: p : \frac{1}{2}N :: S : E$ . Suppose  $S$  an arc in  $DF$ , then  $360 : \frac{1}{2}N ::$  degrees in  $S : E$ ; but deg. in  $S$  (in the circle  $DF$ ) : degrees in  $DG$  (in the circle  $DG$ ) ::  $AD : CD ::$  (by construction)  $360 : \frac{1}{2}N$  in degrees :: degrees in  $S : E$ ; whence the degrees in  $DG = E$ , or the angle  $DAG =$  equation.

[Ib. cor.] for in the syzygies the areas  $ANZ$  and  $eNZ$  vanish; and  $NA$  is  $90^\circ$  when they are in the quadratures; and when in the octants the sine of  $BF = S.90 = 1$ ; therefore,

|      |       |             |         |
|------|-------|-------------|---------|
| :    | 38.3  | .....,..... | 1.58319 |
| Rad. | ..... | .....       | 10.     |
| 1    | ..... | .....       | 0       |

$$S. < A (1^\circ 30') \dots \underline{\quad} \quad 8.41681$$

[Pr. 2, p. 229, and by the demonstration] the angle  $ATN$  will be the distance from the node's true place, by construction. Also the mean motion of the sun in the time  $NA$  : mean motion of the node from the sun in the same time :: (mean annual motion of the sun : mean annual motion of the node from the sun :: (by pr. 1) area of the ellipsis : area of the circle ::)  $TBN : TFN$ ; but  $TBN$  was the mean motion of the sun in the time  $NA$ ; and therefore  $TFN$  is the mean motion of the node from the sun in that time  $NA$ ; and  $FTN$  the angle of the mean motion, or the distance of the sun from the mean plane of the node.

[Ib. cor.] for (Fig. 59) let  $atn = nth = 45^\circ$ , and  $fc \perp$  to  $at$ ; then  $fc$  is the sine of  $fta$ , the  $\Delta$ s  $fcb$  and  $bth$  are similar; therefore  $fc : fb :: bt : bh$ ; and, by composition,  $fh : ct$ , or  $at :: bf$ :

$fc$ ; and, by inversion,  $fc$  is to  $at$ , or  $kt :: fb : fh$ , or by ::  $kH : Hm$ , or  $tk + tH$ .

[Ib. but the fine of] draw  $oz \perp$  to it, then  $oz$  is to  $fc$  in a ratio compounded of (or to  $fb$  or of)  $os$  to  $fg$ , and (the fine of  $orz$  or  $trs$  to the fine of  $fbc$  or  $tbg$ , that is, of)  $ts$  to  $tg$ ; that is, in the compound ratio of  $os \times ts$  to  $fg \times tg$ , that is (because rad. : cosine ::  $2^{ce}$  fine : fine of the double arc, and therefore the rectangle of the fine and cosine being as the fine of the double arc), as fine of  $2otn$  to fine of  $2ftn$ , or radius.

[Ib. sch.] for  $TS : (TS + SK) = TK :: 360^\circ : (360^\circ + 39^\circ,6355) = 399,6355 :: 9,0827646 : 10,0827646$ . Therefore  $TH : TK :: \sqrt{9,0827646} : \sqrt{10,08}$ , &c. And therefore  $TH : HK :: 18,6524761 : 1 :: TS : SH ::$  mean motion of the sun : mean motion of the node =  $19^\circ 18' 1'' 23\frac{2}{3}''$ .

Lastly,  $TK + TH (= 38,22428) : KH (= 1) ::$  rad. : fine of  $1^\circ 29' 57''$  (by cor.).

[Pr. 34, cor. 2, with due regard] for whilst  $p$  moves from  $Q$  to  $F$ , the sum of the areas is comprehended by  $TH$  produced,  $FQ$ , and tangent to  $Q$ ; but, in moving from  $F$  to  $q$ , the line  $Hp$  falls on the other side of the circumference, and the sum of these areas is that comprehended by  $FT$  produced,  $Fq$ , and the tangent to  $q$ ; and the difference of these areas is the semi-circle generated in the time of describing  $QAq$ ; and in the time of describing the whole circumference the whole circle will be generated:

[Ib. cor. 4, that is, as the diameter] Suppose  $Qp$  to be the double distance of the moon from the quadratures;  $MK$  the fine thereof. Then  $Kk : Mp :: MK : \text{radius}$ . And the sum of all the  $Kk$ 's, or the diameter : sum of all the  $Mp$ 's or  $QAq$  :: sum of all the fines : sum of as many radii, or half as many diameters. And, therefore (multiplying the antecedents by  $\frac{Pp}{PG}$ , and the consequents by 2), the sum of all the fines  $\times$

$\frac{Pp}{PG} : \text{sum of as many diameters} :: \text{diameter} \times \frac{Pp}{PG} : \text{whole circumference}$ .

[Pr. 35, by the same increments as the sine of inclination doth, by cor. 3.] For if AEG be double the distance of the nodes from the quadratures, GED will be double the distance of the nodes from the sun; and they both have the same sine.

[Ib. schol.] Here are several particulars about the moon's motion barely laid down, many of them being taken only from observations.

[Ib. p. 235. The force of this action is greater] It is greater the nearer it is, and attracts the moon from the earth, and so dilates the orbit the more. But the farther off, the less force, and the less the moon is drawn from the earth. And at a greater distance she moves slower; and at a less distance faster.

[Ib. farther, I found that the apogee] by cor. 14, pr. 16. b. I. And if  $a$  be the sun's mean distance, and  $a + x$  any other distance, then the motion will be as  $\frac{1}{a+x^3}$  or  $\frac{1}{a^3 - \frac{3x}{a^4}}$ , or  $\frac{1}{a^3} \times 1 - \frac{3x}{a}$ , and the equations, as  $\frac{x}{a}$ , or as  $x$ .

The motion of the sun, that is, the angular motion, is reciprocally as the square of the distance.

[Ib. but if the motion,] let  $v$  be the velocity,  $m$  the mean motion of the sun. Then if  $v$  be as  $\frac{1}{a+x^2}$  or  $\frac{1}{aa} - \frac{2x}{a^3} = m - \frac{2x}{a} m$ , then the equation is  $\frac{2x}{a} m$ . But if  $v$  be as  $\frac{1}{a+x^3}$  or  $\frac{1}{a^3} - \frac{3x}{a^4} = m - \frac{3x}{a} m$ , then the equation is  $\frac{3x}{a} m$ . And the first equation to the second is as  $\frac{2x}{a} m$  to  $\frac{3x}{a} m$ , or as 2 to 3. And hence the greatest equation of the apogee and nodes (sum of all the  $\frac{3x}{a} m$ ): their mean motion ( $m$ ) ::  $2^\circ 54' 30''$ : sun's mean motion; the greatest equations being as the mean motions.

[Ib. p. 236. By the theory of gravity I likewise found] For then the moon being nearer the sun (in all points of its orbit taken together), the sun's force must be greater.

[Ib. p. 237. By the same theory of gravity the action,] For when the sun is in the line of the nodes, its whole force is exerted in moving the moon, and none in moving the plane of its orbit.

[Ib. By the same theory of gravity, the moon's] by cor. 8, pr. 66, b. I.

[Pr. 36, in other positions of the sun,] by the reasoning in cor. 19, pr. 66, l. I. depression of the water at P : ascent at P in the direction PH :: (LM : TM ::) PT : 3PK; and ascent at P in direction PH : ascent at P in direction TP :: (rad. : S.PTK ::) PT : PK; therefore (*ex equo*) depression at P : perpendicular ascent there :: (PT<sup>2</sup> : 3PK<sup>2</sup> ::  $\frac{1}{3}$ PT<sup>2</sup> : PK<sup>2</sup>; but the force PT that depresses P is given, therefore the force to raise the waters at P is always as PK<sup>2</sup>, that is, as the versed sine of 2PTC, or twice the sun's altitude.

Also the effects of these forces (by cor. 14, pr. 66, l. I.) at different distances from the sun are reciprocally as the cubes of these distances; and, therefore,

Cor. 1. The sun raises the water under it to one *Paris* foot and  $11\frac{1}{30}$  inches.

[Pr. 37, but because of the reciprocation] see pr. 24.

[Ib. and the sun's force in — ] (Fig. 60) let A be the position of the moon, and ATP = angle the sun and moon makes at the earth's centre; and let PT represent the force of the sun, which divide into the forces PK, KT, PK, acting in the direction of the moon, increases her force; KT acting in a different direction, diminishes it; and therefore PK — KT (or the difference of the sine and cosine of the angle ATP) is the absolute increase of the moon's force; but this differs not much from the cosine of double the  $\angle$  ATP (in an angle of  $18^{\circ}\frac{1}{2}$  the difference = ,6310189), and in the arcs  $0^{\circ} 45^{\circ} 90^{\circ}$  they perfectly agree. Or thus :

Let qc = moon's force, ca = sun's force (Fig. 61). If the sun is in n, let  $\angle ace = 2 \angle acn$ ; then qc or qb = sum

of the forces,  $cb =$  sun's force, in that position  $=$  cof. of  $\alpha$   
 $<$  acn (Fig. 62).

[ib. But farther; the force of the moon] for let PT be the whole force of the luminary in the place P (or in D), the other side of (the earth or) its parallel; the force DT = the two forces DK, KT. When the luminary is in D, the force to move the water in the direction TP is TK; and the difference of these, or  $(TP - TK =) PK$ , is the difference of the forces in the points P, D; or the absolute force to move the sea at the lat. CTP. But PK is  $\propto \square$  sine of ATP, or as  $\square$  cof. CTP.

Then since  $L + S$  is the whole force when the moon is near the syzygies, and  $L - S$  in the quadratures, therefore  $0,8570327L$  must be substituted for L only in the quantity  $L - S$ , which represents her force in the quadratures.

[Ib. from whence we have] also  $1,017522L$  must be substituted for L in  $L + S$ , which represents her force in the syzygies; and  $0,9830427L$  for L in  $L - S$ , when she is in the quadratures; for these represent her attractive forces in the syzygies and quadratures.

[Pr. 38, as the accelerative gravity] for if the diameters be given, the forces to raise the water will be  $\propto$  absolute forces; and if the absolute forces are given (the spheres will be reduced into similar spheroids, for), the perturbating forces will be as the diameters (for the perturbating force is  $\propto$  PT, Fig. pr. 25). Therefore, if neither be given, the forces to raise the water in the moon and earth are in the compound ratio of the earth to the moon, and the moon's diameter to the earth's diameter, or as 1090 to 100; therefore the water rises to  $93\frac{3}{4}$  feet in the moon.

No notice is taken here of the tides being less by reason of the moon's motion round the earth, than if she stood still in a given position. Since the motion of the tides reaches  $90^\circ$  for every ebbing and flowing, it would seem that the said motion could not be propagated through  $90^\circ$  to its greatest height, in the time the moon stays in the meridian. But (by a computation from pr. 44, 11.) it appears that water will oscillate in a canal of  $90^\circ$  in length in  $\frac{2}{3}$  of an hour; and therefore in that

time the tides will be propagated to that distance, and rise to their highest; but the position of the moon in that time is not sensibly altered; and therefore the tides would rise no higher, though she were always to stand in a given position. But this supposes that there is depth enough of sea to supply water sufficient for the purpose.

[Lem. 1 to pr. 39, to recede towards this side and that side] that is, all the parts of AC to act in a direction towards the sun; the parts of CE directly from it.

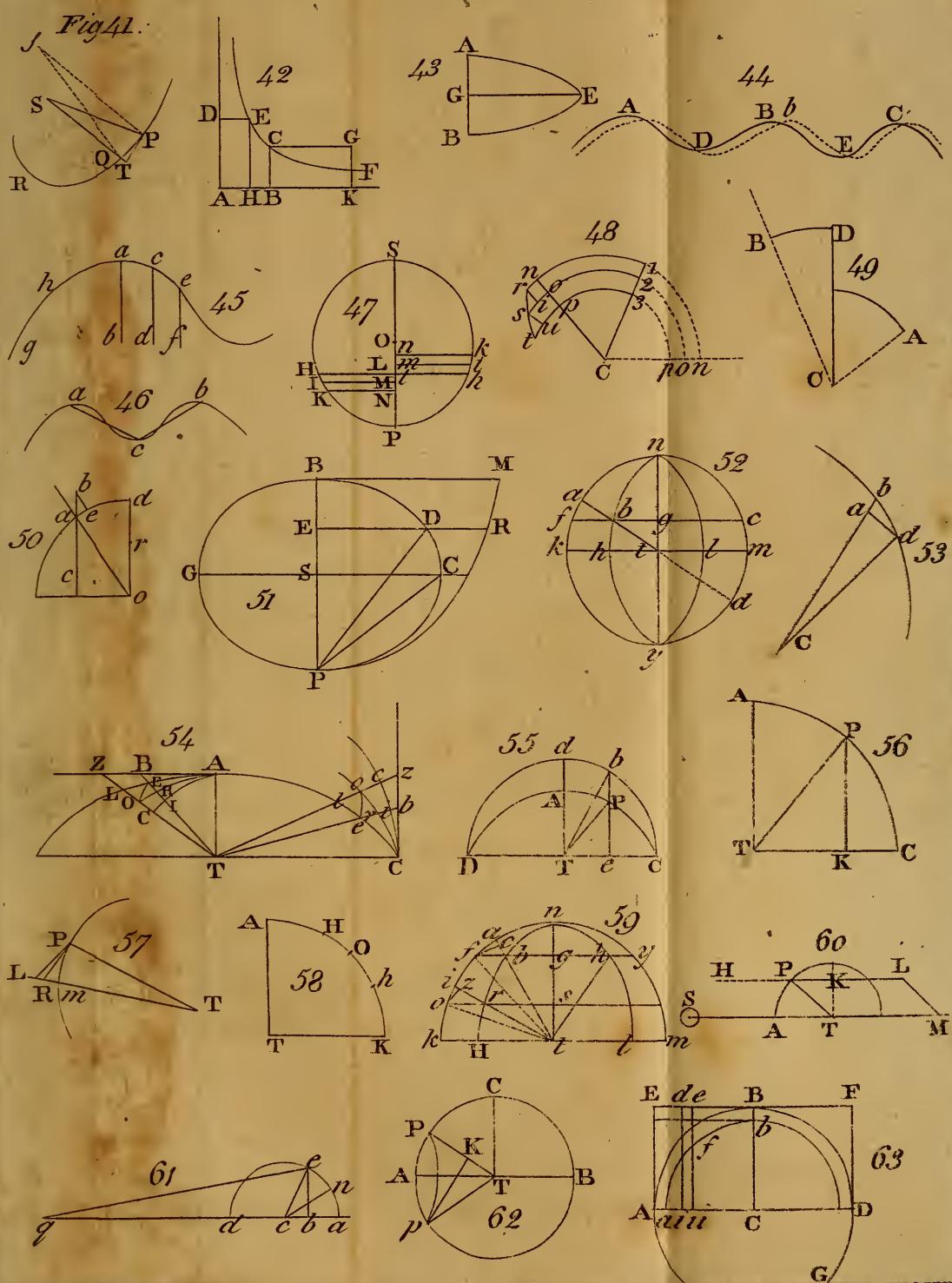
[Lem. 2. The matter in the circumference of every circle IK] by this is meant the circular ring (in the plane of the circle IK) comprehended between the surfaces of the sphere and spheroid.

[Lem. 3, about the same axis] he means about its diameter.

[Ib.] I. Let (Fig. 63) ABD be a spherical surface, and AEFD a circumscribing cylindric surface, and their common thickness the infinitely small line Bb or Aa. Divide AC into an infinite number of equal parts  $=$  in  $= \dot{x}$ , and erect the planes id, ne; then the surfaces of the sphere and cylinder comprehended thereby are equal. Now, flux. mot. cyl. surface : flux. mot. sph. surface :: (as their matter  $\propto$  velocity, and the matter is given, and therefore as) CB  $\times \dot{x}$ , or id  $\times \dot{x}$  : uf  $\times \dot{x}$ . And mot. cyl. surface : mot. sph. surface :: as all the CB  $\times \dot{x}$  : to all the uf  $\times \dot{x}$  :: AD<sup>2</sup> : area of the circle ABDG.

Again; let (Fig. 64) ABD be a sphere, and AEFD its circumscribing cylinder; ECA a cone, all revolving round AD. Divide BC into an infinite number of equal parts  $=$  bd  $= \dot{x}$ . Through the points draw spherical surfaces bgd concentric to ABD, and cylindric surfaces bf terminated at the line EC. Then (by what went before) mot. cyl. surface bdf : mot. sph. surface bdg :: AD<sup>2</sup> : circle ABDG. Therefore mot. all cyl. surfaces, or the solid EBC : mot. all the sph. surfaces, or the sphere : in the same given ratio of AD<sup>2</sup> : to circle ABDG.

Farther; flux. mot. cyl. (Fig. 65) round its axis  $\propto$  flux. mot.  $\propto$  velocity, that is,  $\propto$  ef<sup>2</sup>  $\times \dot{x}$ , or x<sup>2</sup> $\dot{x}$ . And the





whole motion is  $\propto \frac{x^3}{3}$ . And therefore the motion of a cylinder round its axis  $\propto$  length  $\times$  cube of its diameter  $\times$  density; or as its weight  $\times$  diameter.

Lastly (Fig. 66); let EBCA be a cylind. and ECA a cone. Then flux. cyl. mot. : flux. con. mot. :: AE<sup>3</sup>  $\times$   $\dot{x}$  : ba<sup>3</sup>  $\times$   $\dot{x}$  :: CA<sup>3</sup>  $\times$   $\dot{x}$  : Cb<sup>3</sup>  $\times$   $\dot{x}$ , or as r<sup>3</sup> $\dot{x}$  : x<sup>3</sup> $\dot{x}$ . Mot. cyl. round its axis : mot. cone round its axis :: r<sup>3</sup>x :  $\frac{x^4}{4}$  :: (when x = r = CA) 4 : 1. Therefore mot. cyl. : mot. fig. EBC (= difference of mot. of cylinder and cone) :: 4 : (4 - 1 =) 3. Hence it follows,

$$\text{Mot. cyl.} : \text{mot. sol. EBC} :: 4 : 3$$

$$\text{Mot. sol. EBC} : \text{mot. sphere} :: \text{diameter } \square : \text{circle},$$

Therefore, *ex equo*,

$$\text{Mot. cylinder} : \text{mot. of sphere} :: 4 \text{ diam. } \square : 3 \text{ circle.}$$

Or,

$$\text{Mot. sphere} : \text{mot. cylinder} :: 3 \text{ circles} : 4 \text{ diam. square.}$$

II. Flux. mot. cyl. : flux. mot. of a ring (or periphery) (Fig. 67) of the same diameter and matter :: (as velocity  $\times$  flux. matter, as) x<sup>2</sup> $\dot{x}$  : rx $\dot{x}$ . And mot. cyl. : mot. ring ::  $\frac{x^3}{3}$  :  $\frac{rx^2}{2}$  :: (when x = r) 2 : 3. And mot. cyl. : mot. any ring :: 2 cylinders : 3 rings.

III. Let (Fig. 68) ED = z, and let  $\dot{z}$  be given; then  $yz = rx$ . Now flux. mot. ring round C : flux. mot. ring round AB ::  $\dot{rz} : \dot{yz}$  or  $rx : \dot{yz}$ . Therefore mot. ring round C : mot. ring round AB : rz : rx :: (when x = r) AD : AC :: circumference : 2 diameters.

Therefore from the 1st, 2d, 3d, art. it follows, *ex equo*, that the motion of a sphere : motion of a ring round their axes :: as the matter in the cylinder and ring, and 6  $\times$  circumference  $\times$  circle : 24 diam. cube :: or as square of the quadrant arc : to square of the diameter :: ; that is, as matter in the cylinder to matter in the ring, and 61685 : to 100000; or as the matter in the sphere to the matter in the ring, and 925275 to 1000000.

[Pr. 39. (for the reasons above explained,) for (by cor. 2, prop. 30) mean horary motion of the nodes :  $6''\ 35'''\ 16''''\ 36''$  ::  $AZ^2$  :  $AT^2$ . And the sum of all the mean horary motions throughout the year : sum of as many  $16''\ 35'''$ , &c. :: sum of all the  $AZ^2$  : sum of as many  $AT^2$  :: (by lem. 1, prop. 39) 1 : 2; for the diurnal rotation and annual motion of the earth are supposed uniform.

[Ib. But because the plane of the equator,] For, by prop. 30, the disturbing force is as  $3PK$  (and in that prop., the inclination of the planes being small, is not taken notice of); but when the plane of the orbit of P is inclined to the ecliptic, the line  $PK$  is projected into  $pK$ , and the force will therefore be diminished in the ratio of  $PK$  to  $pK$ , or as the radius to the cosine of inclination; for the difference of the distances of P and T from the sun become less in that proportion, and the difference of the forces are diminished in the same ratio.

This being a proposition of great difficulty, depending upon very intricate calculations, which few people are judges of, for this reason, the author's demonstrations have been objected to, and censured by some people as not true. And Mr. Simpson (in his Miscellaneous Tracts), mentioning some of these cavillers, falls into the same notion. He has invented a sort of motion (which he calls *Momentum*) unknown to Sir Isaac Newton, or any body else, and which differs from Sir Isaac's in the ratio of 800000 to 925725. But I never before heard of any motion that was not made up of the quantity of matter and velocity.

Another frivolous objection he makes, is, about the motion of a ring being different from that of the equator; and he tells us, that the motion (*Momentum*) of a ring round its diameter is only half of what it would be when revolving in its plane round the centre. But it is more than half, for it is

as 1 to  $\frac{3.1416}{2}$ , as is demonstrated in art. 3. What he writes afterwards (though he says *it is evident*) is not intelligible. But he concludes at last, in his way, that Sir Isaac Newton has made the precession (by the sun's force) to be but half of what it should be; and as some he mentions had

made the whole proposition erroneous, he modestly ascribes but *two mistakes* to Sir Isaac in this one proposition. But, I believe, that whoever reads the foregoing notes will soon be convinced that his demonstrations are all right; and that all those blunders they tell us of are entirely of their own making, and must be ascribed to themselves only.

[Lem. 4, p. 254. Wherefore if both the quantity of light,] for quantity of light is as the quantity the body receives from the sun directly, and the square of the distance from the body reciprocally; that is,  $\propto$  square of the apparent diameter directly, and square of the body's distance from us reciprocally: therefore the body's distance is as the apparent diameter directly, and  $\sqrt{\text{light}}$  reciprocally.

[Pr. 40, cor. 4] For then the parabola and earth's orbit touch in the vertex, and the area there will be  $= \frac{1}{2}$  velocity  $\times$  distance, or radius, or  $\frac{1}{4}$  parameter; that is,  $= \frac{10000 \times 243.2747}{2}$ , and  $\frac{10000 \times 10.1364\frac{1}{2}}{2}$  for the diurnal and

horary motion. The rest is plain, from pr. 14, l. 1.

[Lem. 5.] (Fig. 69.) For the ordinates AH, BI, CK, &c. put a, 2a, 3a, &c. and take the differences as follow :

|    |    |    |    |    |    |
|----|----|----|----|----|----|
| a  | 2a | 3a | 4a | 5a | 6a |
| b  | 2b | 3b | 4b | 5b |    |
| c  | 2c | 3c | 4c |    |    |
| d  | 2d | 3d |    |    |    |
| e  | 2e |    |    |    |    |
| f. |    |    |    |    |    |

Now to find any ordinate, as 4a, we have  $4a = 3a - 3b$   
 $= 2a - 2b = a - b$   
 $- 2b + 2c = - 2 \times b + 2 \times c$   
 $+ d = a - 3.b + 3.c - d.$   
 And so of others.

CASE 1. If we substitute for p, q, r, &c, their equals, we shall have,

$$RS = a$$

$$+ b \times - SH$$

$$+ c \times - SH \times - \frac{SI}{2}$$

$$+ d \times - SH \times - \frac{SI}{2} \times \frac{SK}{3}$$

$$+ e \times - SH \times - \frac{SI}{2} \times \frac{SK}{3} \times \frac{SL}{4}$$

$$+ f \times - SH \times - \frac{SI}{2} \times \frac{SK}{3} \times \frac{SL}{4} \times \frac{SM}{5}, \text{ &c.}$$

Now let RS fall upon any ordinate, as, for example, on LD; then SH, SI, SK, SL, &c. will be = LH, LI, LK, o, &c. = 3, 2, 1, o, &c. respectively; which being written in the aforesaid series, we have

$$RS = a - 3.b - 3 \times \frac{-2}{2}c - 3 \times \frac{-2}{2} \times \frac{-1d}{3} +$$

o e = a - 3.b + 3.c - d, which, by what went before, is the value of the ordinate DL; and since this holds generally, it is plain this series will give the value of RS wherever it falls.

CASE 2. Let x represent any base HS, y any ordinate RS. Assume  $y = A + Bx + Cx \times \overline{x-P} + Dx \times \overline{x-P} \times \overline{x-Q}$ , continued to as many terms as there are ordinates, which suppose four; P, Q, R, &c. being respectively = HI, HK, HL. Then, putting AH, BI, CK, DL, successively for y and o, HI, HK, HL, for x, we shall have from the general equation AH = A, BI = A + BP, CK = A + BQ + CQ  $\times \overline{Q-P}$ , DL = A + BR + CR  $\times \overline{R-P}$  + DR  $\times \overline{R-P} \times \overline{R-Q}$ , that is, from the figure, AH = A = a.

$$BI = A + B \times HI.$$

$$CK = A + B \times HK + C \times HK \times IK.$$

$$DL = A + B \times HL + C \times HL \times IL + D \times HL \times IL \times KL.$$

Whence, by Subtraction,

$$AH - BI = - B \times HI.$$

$$BI - CK = - B \times IK - C \times HK \times IK.$$

$$\begin{aligned} CK - DL &= -B \times KL - C \times \overline{HK} + \overline{IL} \times KL = D \\ &\quad \times HL \times IL \times KL. \end{aligned}$$

Then, dividing by the coefficients of B,

$$\frac{AH - BI}{HI} = -B = b.$$

$$\frac{BI - CK}{IK} = -B - C \times HK = 2b.$$

$$\begin{aligned} \frac{CK - DL}{KL} &= -B - C \times \overline{HK} + \overline{IL} - D \times HL \times IL \\ &= 3b. \end{aligned}$$

Again, by Subtraction,

$$b - 2b = C \times HK.$$

$$2b - 3b = C \times IL + D \times HL \times IL. \text{ And, dividing,}$$

$$\frac{b - 2b}{HK} = C = c.$$

$$\frac{2b - 3b}{IL} = C + D \times HL = 2c. \text{ Also}$$

$$c - 2c = -D \times HL; \text{ and}$$

$$\frac{c - 2c}{HL} = -D = d.$$

Thus the coefficients A, B, C, &c. are determined for four ordinates, and they are found the same way, if more ordinates are given. Then taking x at pleasure, as suppose = HS, then RS or y will be  $a + bx + cx \times \overline{x - P} + dx \times \overline{x - P} \times \overline{x - Q}$ , &c. =  $a + b \times p + c \times p \times IS + d \times p \times IS \times SK$ , &c. =  $a + bp + cq + dr$ , &c.

[Cor. lem. 8, for then  $AC : A\delta (= AI - \delta I) :: A\mu C : A\delta dy - d\mu x (= AI\mu y - \delta I\mu X)$ , and ::  $ASC : AS\delta$  (Propor. 10) ::  $(A\mu C + ASC) = ASC\mu A : (A\delta dy - d\mu x + AS\delta = AS\delta dy A - d\mu x =$  (because the  $\Delta s x\delta\mu$  and  $\delta S\mu$  are equal)  $AS\delta dy A - d\mu x + x\delta\mu - S\delta\mu = AS\delta\mu dy A - S\delta\mu = AS\mu dy A$  : time of describing  $A\mu C$  : time of describing  $Ay\mu$ .]

[Ib. Schol.] For Bn will cut the chord AC in a point a little beyond E (towards C), suppose at e. Now the area  $AEX\mu A$  to the area  $ACY\mu A$  is in something a greater ratio than AE to AC, as suppose as Ae to AC, more near than before. Therefore  $ASEX\mu A : ASCY\mu A :: Ae : AC$ . Whence

$\text{ASBX}\mu\text{A}$  (or  $\text{ASEX}\mu\text{A}$ ) :  $\text{ASCY}\mu\text{A} :: \text{Ae} : \text{AC}$ ; or  $\text{ASBY}\mu\text{A}$  :  $\text{ASCY}\mu\text{A} :: (\text{Ae} : \text{AC} ::)$  time in  $\text{AB}$  : time in  $\text{AC}$ , very near:

[Lem. 10, to the triangle ASC, that is,]  $\text{AC}\mu\text{A} : \text{ASC} :: \text{AC} \times \frac{2}{3}\text{M}\mu : \text{AC} \times \frac{1}{2}\text{SM}$  (because the  $\angle$ s SMA and  $\text{AI}\mu$  are equal, and  $\text{M}\mu = \text{I}\mu$ )  $:: \text{AC} \times \frac{1}{2}\text{M}\mu : \text{AC} \times \text{SM} :: \text{MN} : \text{SM}$ . And, by composition,  $\text{ASC}\mu\text{A} : \text{ASC} :: \text{SN} : \text{SM}$ .

[Lem 11. Subduplicate proportion of 1 to 2,] and therefore  $= \frac{\text{AC}}{\sqrt{2}}$ . And the arc in half the time  $= \frac{\text{AI}}{\sqrt{2}}$ . And  $\frac{\text{AI}^2}{2} \div 2\text{SP} = \frac{\text{AI}^2}{4\text{SP}} =$  space described in descending.

[Prop. 41, and BE, by lem. 11, is a portion,]  $tS$  and this hypotenuse (mentioned in the prop.) are the distances of the earth and comet from the sun. Now BE to  $tV$  is compounded of BE to the part of the hypotenuse (projected into BE), or of BS to the whole hypotenuse, and of that part of the hypotenuse to  $tV$ ; that is, as the gravitating forces at the comet and earth (for these lines would be described, by falling bodies in equal times, by these forces); that is, as  $\text{St}^2$  to hypotenuse square : and therefore  $\text{BE} : tV :: \text{BS} \times \text{St}^2 : \text{hypotenuse cube}$ .

Farther (Fig. 70); because of the immense distance of S, any of the points B, i,  $\mu$ , are nearly in the curve of the parabola, and  $\mu$  the vertex, and  $\mu p$ ,  $i\lambda$ ,  $q\sigma$ , nearly parallel. Therefore  $\angle I\mu i =$  (by reason of the rectangle  $Ii\mu\lambda$ )  $\angle p\mu L$ , or  $= S\mu L$ . Therefore  $\mu I$  is the diameter to the vertex  $\mu$ ; also  $\xi\sigma =$  (by construction)  $3S\sigma + 3i\lambda = 3S\sigma + 3I\mu = 3S\sigma + 3\mu p =$  (because  $2O\sigma = 3\mu p$ )  $3S\sigma + 2O\sigma$ ; and therefore  $\xi O = 3S\sigma + 3O\sigma = SOS$ . But  $\mu O : I\mu :: p\sigma : Ip$ . But  $\mu I$  is bisected by  $i\lambda$ ; therefore  $Ip = 2i\lambda$ ; and  $p\sigma = I\lambda = \frac{1}{2}Ip$  (since  $I\sigma = 3I\lambda$ , by construction); therefore  $\mu O = \frac{1}{2}\mu I$ . Whence (by lem. 8) a line drawn from  $\xi$  divides the chord AC nearly as the time; whence BE drawn towards  $\xi$  is rightly drawn. But (by construction) new BE : former BE ::  $\text{BS}^2 : \overline{S\mu + \frac{1}{3}i\lambda}^2 ::$  as the gravitating force at the

distance  $S\mu + \frac{1}{3}i\lambda$  : to gravitating force at the distance B or  $\mu$  :: space fallen through at the distance  $S\mu + \frac{1}{3}i\lambda$ , in the time of describing the arc  $\frac{1}{2}A\mu C$  : space fallen through at B or  $\mu$ , in the same time. But former BE equal space fallen through at  $\mu$ . Therefore new BE = space fallen through at the distance  $S\mu + \frac{1}{3}i\lambda$ , in half the time the comet describes the arc intercepted between TA,  $\tau C$  = (by lem. 11)  $i\lambda$ , or this new BE, nearly; which, therefore, is rightly determined nearer than before.

Because the distance of the sun and comet is something more than the hypotenuse of the triangle, whose base is  $S\mu$ ; and perpendicular IO (because O is only a point in the chord, which should be in the arc), therefore he supposes that distance = hypotenuse (whose base is  $S\mu + \frac{2}{3}i\lambda$ , and perpendicular IO, that is) = DO. Also  $MP : X ::$  (by construction)  $\sqrt{St} : \sqrt{OD} ::$  (by cor. 6, pr. 16, I.) velocity of the comet at the distance DO : velocity thereof at distance St :: space described (uniformly) at the distance DO, and in the time of describing the arc ABC : space described (uniformly) at the distance St, in the same time, which is equal X, by construction; and therefore  $MP =$  space described (uniformly) at the distance DO (in the time that the arc ABC is described) = (by lem. 10, cor.) to the chord of that arc.

The points e, a, c, g; and  $\varepsilon, \alpha, \chi, \gamma$ , being found out as E, A, C, G, before, is with an intent to find, at last,  $MP = MN$ , or  $AC = AG$ . Wherefore several of them being thus found, and a simple curve (viz. a circle) drawn through these points, finds the points Z and X, where those lines would be equal; for the nearer they approximate to equality, the nearer they come to the true points of the comet's orbit.

Farther; velocity of a comet at Q (in a parabola) : velocity of a comet at Q (in a circle) (by cor. 7, prop. 16, I.) ::  $\sqrt{2} : 1 ::$  (nearly as)  $\frac{4}{3} : 1$ . Also velocity at Q (in a circle) : velocity at t (in a circle) (by cor. 6, pr. 4, I.) ::  $\sqrt{St} : \sqrt{SQ}$ . Therefore, *ex equo*, velocity of the comet at Q, in a parabola : velocity of the earth in its orbit at t ::  $\frac{4}{3}\sqrt{St} : \sqrt{SQ}$ . But velocity of the comet at Q : velocity of the earth at t :: (nearly as)  $AC : T\tau$ . Therefore  $AC : T\tau :: \frac{4}{3}\sqrt{St} : \sqrt{SQ}$ ; or

$AC : \frac{4}{3}T\tau :: \sqrt{St} : \sqrt{SQ}$ . As it is by construction (for AC is to  $\frac{4}{3}T\tau$  in the reciprocal subduplicate ratio of SQ to St, and not in the direct subduplicate ratio, as is falsely printed); therefore Q is (nearly) in the chord of the parabola, and B a point of the comet's orbit, nearly.

Lastly; if  $MP = MN$ , or  $AG = AC$ ; then  $Yb : YB :: Yc : YE :: ac : AC :: \text{velocity in } b : \text{velocity in } B :: \sqrt{SB} : \sqrt{Sb}$ . But if  $Sb = SB$  and MP or AG invariable, it will be  $Yb : YB :: ac \text{ or } AG : AC$ , when the point G falls in CY. Therefore, universally,  $Yb : YB :: AG \times \sqrt{SB} : AC \times \sqrt{Sb} :: MP \times \sqrt{SB} : MN \times \sqrt{Sb}$ , to find the point b true.

[Ib. p. 277, may be seen in the following table.] The places of the comet in this and all the other tables are the geocentric longitudes and latitudes of the comet,

[Ib. p. 283. And thinking, &c.] In Plate 18, while the comet passes through I, K, L, M, &c. the earth passes from between P and K towards G: Whence the comet will move swift through LKL and NOP; slower through LBM and QRS, as is said in the foregoing p. 282.

[Ib. p. 285, as about 6 to 1000,] for, by p. 274-5,  $\frac{\text{lat. rect.}}{4} =$   
6, nearly; when earth's dist. = 1000.

[Ib. p. 290, I (Fig. 71) found, that at the height, &c.] Let SA = rad. earth = 20950000 English feet, nearly, AB = 850, AF = AS, and the densities AH and BI are as 33 to 32. But (by cor. 2, p. 22, II.)  $Aa - Bb : Aa - Ff :: \text{thin} : \text{thnz}$ ; that is,  $AB \times SF$ , or  $2AB \times SA : AF \times SB$ , or  $SB \times SA :: \text{thin} : \text{thnz}$ ; or  $2AB : SB :: \text{thin} : \text{thnz}$ ; that is,  $1 : 12324 :: \text{thin} : \text{thnz} ::$  (by schol. pr. 86, hyperbola) log. St — log. Su : log. St — log. Sz :: ,013364 : 1.518514 — log. Sz. Whence log. Sz = — 164.820578; and SZ or FN = (,0<sup>163</sup> ciphers) 66 +. Wherefore FN : AH :: ,0<sup>163</sup> 66 : 33 ::, or as  $2 : 10^{165} :: 1 : 10^{164} ::$  density in F : density in A. But the diameter of Saturn's orb is = 191 times the semi-diameter of the orb's magnus =  $191 \times 20000$  radii of the earth =  $191 \times 20000 \times 20950000 \times 12$  inches =  $96 0^{13}$  + inches. And a sphere of 1 inch diameter : to this sphere :: as  $1 : \sqrt[3]{960^{13}} :: 1 : 880^{13} +$ ; but  $10^{164} > 80^{14}$ . Therefore, &c.

[Pr. 42.] the correction of the orbit here given is computed by the Rule of False; 1st, for the longitude of the node, by operation 1 and 2; and, 2dly, for the inclination of the orbit, by operation 1 and 3.

I. for the longitude of the node; since (by construction) time of describing D : time of describing E :: D : E :: G : 1; and (time of describing E, nearly =) time B : time A :: 1 : C; therefore (*ex equo*) time of describing D : time A :: G : C; therefore G — C is the error of the times between the first and second observations, for the given longitude of the node K, and the given inclination of the orbit I. Again;

Time of describing d : time of describing e :: d : e :: g : 1; and time of describing e (nearly =) time B : A :: 1 : C; therefore time of describing d : time A :: g : C; and g — C is the error of time between the two first observations, for the longitude K + P, and inclination I; therefore, by the Rule of False, by two positions (which says, diff. er. : diff. positions :: either er. : correction of its position), G — g : G — C

:: P :  $\frac{G - C}{G - g}$  P = the correction of K; or the computation may be thus: since to the supposition K and K + P

the errors between the first and third observations in these two cases are T — S, and t — S, therefore T — t : T — S ::

P :  $\frac{T - S}{T - t}$  P = the same correction of K; therefore mP

(is to be) =  $\frac{T - S}{T - t}$  P =  $\frac{G - C}{G - g}$  P; and therefore mT —

mt = T — S; and mG — mg = G — C.

II. For the inclination of the orbit, it is plain T — S = first error of the times between the first and third observations to the given node K, and inclination I; and r — S = error between the first and third observations to the node K, and inclination I + Q. Therefore (by the Rule of False) T — τ : T — S :: Q :  $\frac{T - S}{T - \tau}$  Q = correction of I. Again; this time of describing δ : time of describing ε :: δ : ε :: γ : 1; and (time of describing ε = nearly) time B : A :: 1 : C: ergo time of describing δ : A :: γ : C; therefore γ — C = error

of the times between the first and second observations to the node K and inclination I + Q; and G - C was the error between the first and second observations to the node K and inclination I; therefore (as before)  $G - \gamma : G - C$

$$\therefore Q : \frac{G - C}{G - \gamma} Q = \text{correction of } I; \text{ therefore } nQ \text{ (must be)}$$

$$= \frac{T - S}{T - \tau} Q = \frac{G - C}{G - \gamma} Q; \text{ and } nT - n\tau = T - S, \text{ and}$$

$$nG - n\gamma = G - C. \text{ Lastly; from (I. and II.) } 2T - 2S = mT - mt + nT - n\tau; \text{ and } 2G - 2C = mG - mg + nG - n\gamma; \text{ therefore the numbers } m, n, \text{ are rightly found.}$$

[Ib. and, lastly, if in —] because the longitudes in the first and second orbits are K and K + mP, therefore the *latus rectum*s will be R and  $R + \overline{r - R} \times m$ ; and because, in the first and third orbits or planes thereof, the inclinations are I and I + nQ, therefore the *latus rectum*s will be R and  $R + \overline{\varrho - R} \times n$ ; or  $R + \overline{r - R} \times m$ , and  $R + \overline{r - R} \times m + \overline{\varrho - R} \times n$ .

Or thus; if the longitude K be increased (in op. 1, 2) by P, then (because the distance of the perihelion from the sun is increased accordingly) the *latus rectum* will be increased by  $r - R$ ; and if the longitude be increased by mP, the *latus rectum* will be increased by  $mr - R$ . And, for the same reason (in op. 1, 3), if the inclination be increased to nQ, the *latus rectum* will be increased to  $\overline{N\varrho - R}$  (for at the increase Q the increase of the *latus rectum* is  $= \varrho - R$ , since the whole is  $\varrho$ , or  $R + \overline{\varrho - R}$ ); therefore if both longitude and inclination be increased; the *latus rectum* will be increased in both these respects, and will become  $R + mr - mR + n\varrho - nR$ .

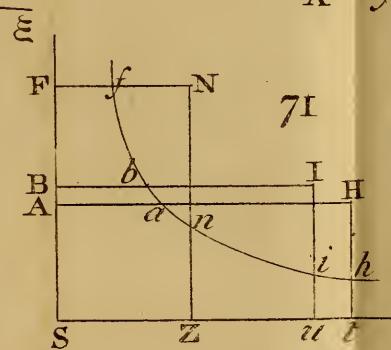
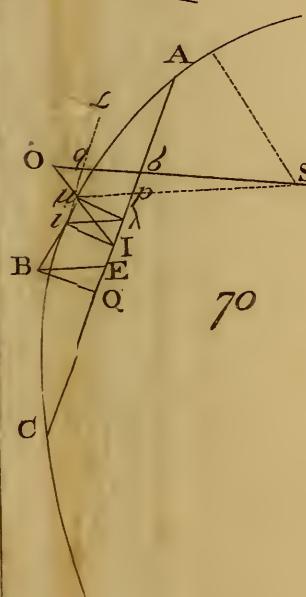
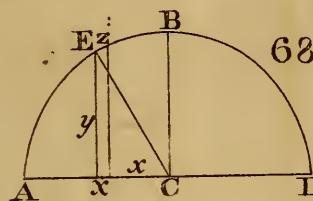
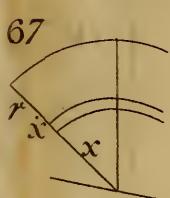
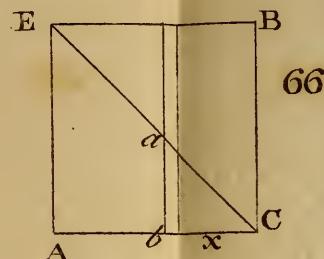
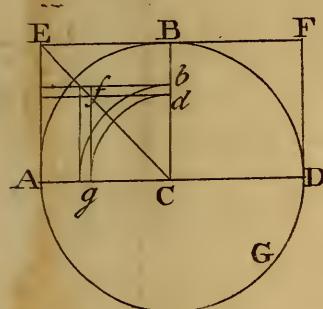
Also when K becomes  $K + P$ , the transverse becomes

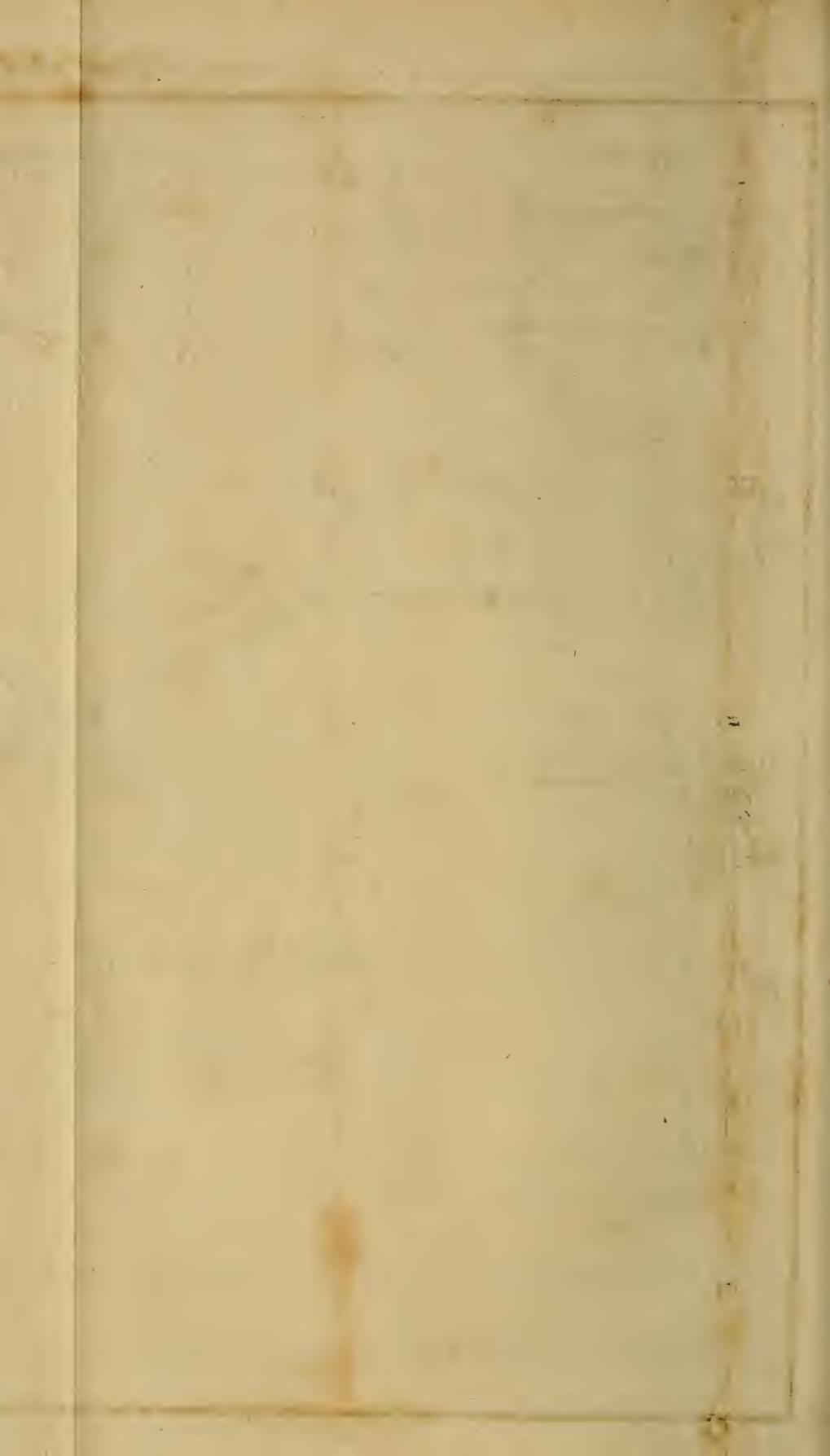
$$\frac{1}{L + 1 - L} \text{ or } \frac{1}{1}; \text{ also when K becomes } K + mP, \text{ the transverse becomes }$$

$$\frac{1}{L + m.1 - L}; \text{ for the same reason, when}$$

$$I \text{ is increased to } I + nQ, \text{ the transverse is } \frac{1}{L + n. \lambda - L}.$$

*Fig. 64.*





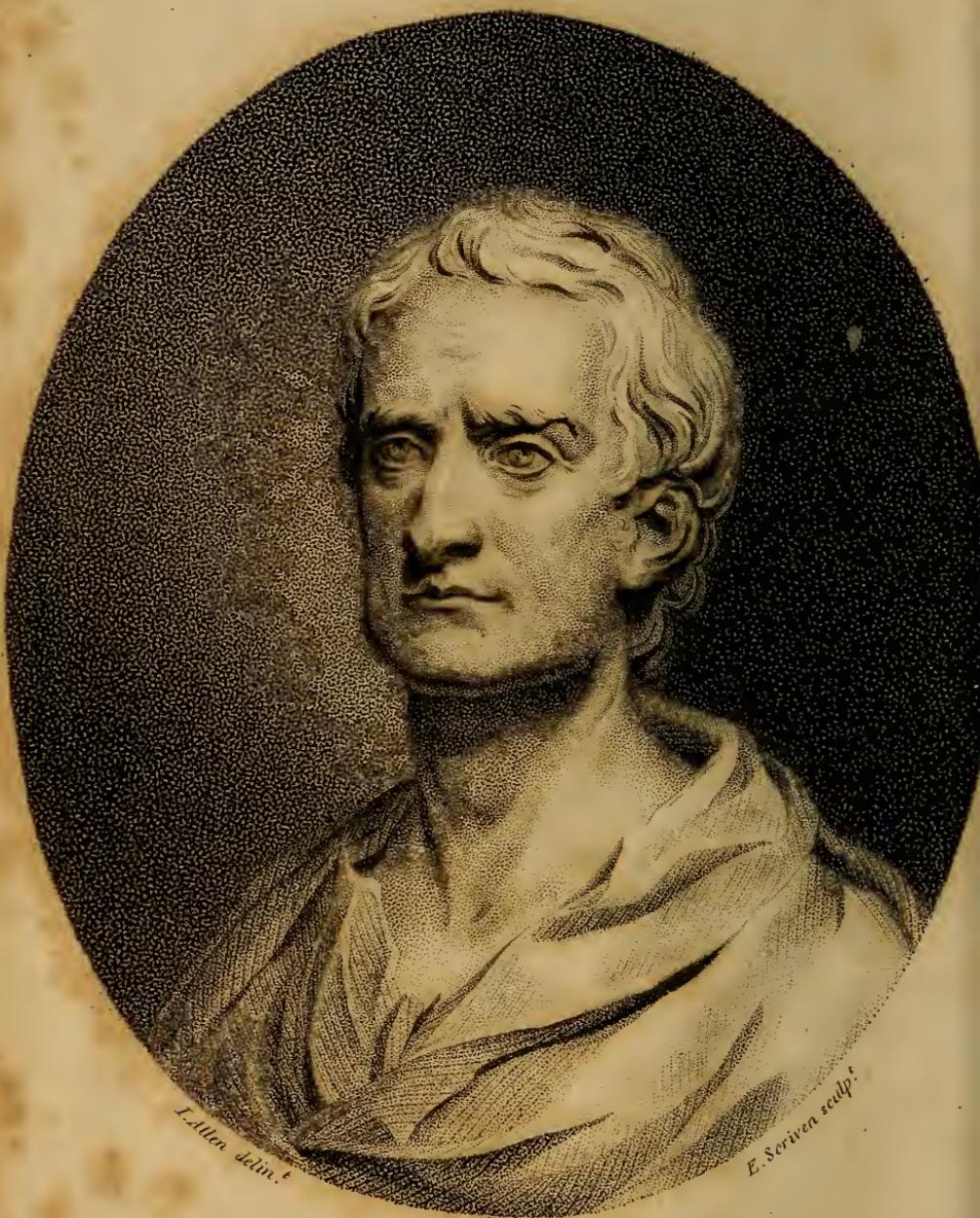
Therefore when both longitude and inclination are increased, the transverse will then be  $\frac{1}{L + m\lambda - mL + n\lambda - nL}$ .

[Ib. sch. p. 307, the comet] for, by p. 285, comet's dist. : earth's dist from  $\odot :: 6 : 1000 ::$  dist. comet in  $\odot$  diameters : (earth's dist. in  $\odot$  diam.  $=$ ) 109  $\odot$  diam.; therefore comet's dist. from the sun's centre  $= \frac{654}{1000} = 654$  diams. and from its surface  $= ,154$  diams.  $= \frac{1}{6}$  sun's diameter.

*End of the Comment.*







# SIR ISAAC NEWTON.

*From a Bust at the Royal Observatory, Greenwich -*

*London. Published Nov 10, 1802, by H.D. Symonds Paternoster Row.*

A

# DEFENCE

OF

## SIR ISAAC NEWTON,

AGAINST THE OBJECTIONS THAT HAVE BEEN MADE

TO

*Several Parts of the Principia.*

---

Raro antecedentem scelestum,  
Deseruit pede pœna clando.     HOR. LIB. III.

---

THIS incomparable Treatise being written in a concise style, and in the synthetic method, and being upon subjects quite new and untouched before, the generality of readers could make little of it. As it contained a new system of philosophy, built upon the most sublime geometry, the greatest mathematicians were obliged to study it with great care and attention, and few became masters of the subject; so that for a long time it was little read. But, at last, when the value of it became more known, it gained universal approbation, and the whole world stood amazed at the numberless new discoveries contained therein. And, upon account of its universal agreement with all the phænomena of nature, it was adopted as the only true system by all, except some few that, through envy or ignorance, were bigoted to some other scheme.

It is true, many things therein depend upon very difficult mathematical computations not easily comprehended by every reader; and this has given occasion to some people to question, or even deny, the truth of several propositions laid down in this work; and have therefore given computations of their own, different from his, which they assert for truth, and thereby, as they think, proving his to be false. I shall, therefore, spend a little time in answering the most material of these objections, and shall shew that these principles objected against

are all true in the sense Sir Isaac meant them ; and that these objections are only the effect of their own ignorance or inattention.

J. Bernoulli (tom. IV. p. 340) thinks Newton's demonstration of cor. 4th to the laws is imperfect, where he says, *only the action of two bodies is considered* ; and lays down a method by which he thinks it may be fully demonstrated ; but at last leaves it undemonstrated himself, and tells you that this is the foundation, but leaves the calculation for others to make out.

But Sir Isaac Newton, after he had shewn the truth of the cor. in two bodies, shews that the common centre of gravity of these two bodies and a third will either be at rest, or move uniformly in a right line ; and from thence that the centre of gravity of *any two* (in a system of bodies) suffers no change of state by their actions upon one another. From whence it follows, that in the whole system all these actions (composed of the action of *every two*) can induce no change ; and if the actions of *any two* make no difference in the centre of gravity of the whole, separately considered, then the actions of *every two* considered collectively can make no difference. This is certainly so, though delivered in a few words : Sir Isaac was not writing to children.

The same Bernoulli (p. 347) objects against the 10th pr. b. II. for not being solved in a way general enough to please him. Here Sir Isaac has solved the problem according to that law of resistance which really exists ; but could not spare time, or fill up his work with useless enquiries, or things of no consequence. His noble structure, the *Principia*, is all gold ; but he has left the dross for such authors as these, who are fond of any thing, and will serve well enough for building up their systems. This author in particular is always depreciating Sir Isaac, and extolling his own performances ; though they are long, tedious, and laborious to the last degree, and often false ; although he has the advantage of the analytical and fluxional methods, and seems to be ignorant that the method of analysis by which he (*Bernoulli*) solves these problems is incomparably shorter than the method of composition in

which Sir Isaac has written. Had he written the Principia in the analytic way, or according to the method of Fluxions, a science then utterly unknown to the world, it could not have been read by any man living. By the same way of reasoning, this boasting mathematical bully may as well condemn *Euclid's Elements*, or *Apollonius's Conics*, because most of the propositions may be demonstrated incomparably shorter by common algebra.

This Author seems to have a particular spite against the English mathematicians, being always carping and criticising. His works are full of invectives against Sir Isaac Newton for his great discoveries, which this low critic is unsuccessfully endeavouring to imitate. When one person has another man's works to look through, he must be a great blunderbuss that cannot make some small additions : a man placed upon another's shoulders will see farther than his supporter ; yet in many cases he has disimproved Sir Isaac Newton. Sir Isaac had a work entirely new to execute, in a short and general way, and could not take notice of every bauble : this author has had nothing to do but to imitate. The regular works of the one will live through all ages ; but the confused chaos of the other will sink into everlasting oblivion.

The same author (p. 484) also objects against the demonstration of the 36th prop. b. II. For, says he, *since in Newton's cataract there is no compression of the water in any place, nor against the sides of the cataract, the external water pressing inwards must disturb the cataract, and mix with it ; and, therefore, the Newtonian explication being contrary to the laws of Hydrostatics, cannot subsist.* Yet what Newton says (case 1) might have satisfied him. His words are ; *For let the ice in the vessel dissolve into water, yet will the efflux of the water, as to its velocity, remain the same as before. It will not be less, because the ice, now dissolved, will endeavour to descend : it will not be greater, because the ice, now become water, cannot descend without hindering the descent of other water equal to its own descent.* But this author is eternally cavilling at every thing ; and all his whole section on this subject, consisting of about 100 pages, where he has tossed this

problem about into all forms, is nothing but a heap of absurd, inconsistent stuff, neither agreeable to theory nor experiment. So we shall leave him here drowned in a *gurges* of his own contriving.

In the demonstration of prop. 47, b. II. Sir Isaac Newton assumes for a principle, that the parts of the air have a motion excited in them, by some cause or other, according to the laws of an oscillating pendulum. But *Euler* has found fault with him for making such an hypothesis; and has himself assumed some different hypotheses, from which he has pretended to demonstrate that the motion of the particles of air are moved according to such hypotheses; and that his demonstration for his hypothesis is as valid as Newton's is for his. But Sir Isaac Newton's hypothesis is more than an assumed one, for it is true in fact; for any tremulous or vibrating body from which sound is propagated is known to vibrate according to this law, assumed by Newton; and the like motion is communicated to the particles of air. Therefore he has rightly assumed that the medium has such a motion excited in it; and consequently that it will continue to move according to that law. But there is no such law of a vibrating body as *Euler* assumes, and therefore his demonstrations come to nothing. To find the velocity of sound is a very subtle problem; and they that would see it truly resolved may find it in my book of Fluxions, prob. 23, sect. 3.

In b. II. prop. 52, Sir Isaac Newton has investigated the properties of vortices, in order to see whether the celestial motions can be accounted for thereby; and he has shewn that the celestial bodies are not carried round in vortices. But *J. Bernoulli*, a person of eternal contradiction, has undertaken to defend the notion of vortexes, though shewn to be absurd and impossible. Let him frame what hypothesis he will for the planets, he must find vortexes meeting one another for the comets, that move all manner of ways. Therefore, to clear this point, he must have his vortexes made of such fluid matter as has neither friction nor resistance, and that one part be penetrable to another part; that is, such a fluid as cannot be found in the world. Such is the blinded,

bigoted, prejudice of some people, that the clearest demonstrations cannot cure their madness. If any body asks one of this stamp how the planets move round, they presently answer, in vortexes. But ask them what causes the vortexes to move round, here they are at a stand, and have nothing to say, but that they move by their own nature: but they might as well have said at first that the planets move round by their own nature, if they can give no better account of them, and so save the labour of constructing these useless vortices.

Another author that is *vortex-mad*, is *Euler*, and he seems to go beyond any of the rest, for he cannot account for the rising of the tides without vortexes; but he has not shewn us by what extraordinary mechanism, or invisible wheelwork, his vortices are constituted, so as to be able to produce the tides, or cause the motion of the comets, since these vortices must all run counter to one another, and penetrate one another, and yet miraculously preserve their motions entire.

This idle notion was first introduced in the time of ignorance, upon the supposition of a *fuga vacui*, or that nature abhorred a *vacuum*; but, by our better acquaintance with the nature and properties of body, and the laws of motion, we now know that the operations of nature cannot be performed in a *plenum*, and therefore a *vacuum* is absolutely necessary.

If we had known of no celestial bodies but the planets moving all one way, the supposition of a vortex to carry them about, clumsy as it is, might have passed for possible; but one would have thought that the comets' moving all manner of ways would have cured this delirious notion, and have taught them the impossibility of such a scheme: but these authors, these defenders of vortices, are so hardy, that they are not at all afraid of an absurdity or a contradiction. These things do not affect or touch them in the least; but they go on unconcerned in their usual way, though contrary to all the laws of nature.

Did I say that the theory of vortices had its original from the principle of a *fuga vacui*? I did. And the principle of a *fuga vacui* had its rise from the phenomena of water rising

in pumps and siphons, as in the *Torriceilian experiment?* Here, for want of trying such experiments to their full extent, the principle of a *fuga vacui* was assumed by these philosophers as the genuine cause of these effects; so that, upon no better a footing than this, though the true cause of these effects (the external pressure of the air) be now perfectly known, yet with these bigoted people the same false principle still exists.

These philosophers (if they can be called such) are endeavouring, contrary to the nature of things, to find out the most complicate causes for explaining the phænomena, instead of seeking the simplest causes; and they seem utterly to reject all simple causes, which are the greatest beauty of nature; for if their brains were not turned round in a vortex, they could never prefer these complex vortical schemes before the simple doctrine of projectile and centripetal forces. Such Philosophers!

Sir Isaac Newton had shewn, in the schol. of prop. 14, b. III. that the aphelions of the interior planets move a little *in consequentia* (by the actions of Jupiter and Saturn) in the sesquiplicate ratio of their distances from the sun; but *Bernoulli* says this is false, for it holds not true in Saturn; and yet Sir Isaac tells him expressly it is the inferior planets that observe that law. So little do some men care what they write, if they can only put on an air of contradiction.

In finding the proportion of the axis and diameter of the earth, prop. 19, b. III. Sir Isaac assumes them to be as 100 to 101, and from thence finds the excess of weight in one above the other to be  $\frac{4}{505}$  parts. Therefore he makes this

proportion  $\frac{4}{505} : \frac{1}{100} :: \text{the force } \frac{1}{289} : \frac{1}{229}$ , the true excess of height. Yet some people have objected against this as a wrong way. But herein they are much mistaken; for if any force be increased, the effect will be increased in the same ratio; and therefore increasing the centrifugal force here will increase the difference of the heights in the very same proportion, as these quantities are very small. Upon this

account, some people have unnecessarily run into long calculations, and only came, at last, to the same conclusion.

And the same argument holds good against such as have objected against finding the height of the tide (cor. prop. 36, b. III.) by the rule of proportion, comparing it with the centrifugal force; for all forces of whatever kind will produce effects proportional to the quantities of these forces; and therefore he has rightly found the height of the tide by that proportion.

But Sir Isaac Newton's explanation of the tides (prop. 24, b. III.) does not please *Euler*, though he accounts for every circumstance thereof. He thinks ascribing these effects to the actions of the sun and moon is recurring to *occult causes*, and therefore he had rather recur to *vortexes* for the explanation thereof, the notion of which has been confuted over and over. He denies the gravitation of bodies towards one another, because he cannot discover the cause of gravity; and therefore he will not allow it to have any thing to do with the matter, as being an occult quality. But he recurs to a principle that is more than occult, his incomprehensible vortices, which he thinks the tides are raised by, though he has not attempted to explain in what manner his vortices can do it.

He says, Sir Isaac Newton's account of the tides is not sufficiently explained, that any certain judgment can be formed whether it is true or false; that he did not explain the phænomena of the tides, but only darkened them. But certainly this man does not *know* what he writes, or does not *care* what he writes. Sir Isaac explained all the principal phænomena thereof, and has shewn that his theory is agreeable to observations both in nature and quantity; and all matters of less note may be easily deduced from his general theory by any intelligent person. But Sir Isaac did not choose to erect any *pontes asinorum*.

This author values himself very much for rejecting the principle of *attraction* established by *some Englishmen*, because it is an occult quality. This profound Philosopher will either know every thing or nothing; he will make no use of the

forces of gravity, because he knows not the cause of gravity. For the very same reason he would want to know the cause of that cause ; and so he must know every thing in the whole chain of causes and effects, or he cannot be satisfied. This odd temper in some men arises from the pride of the human mind, which attempts to soar above its sphere. They disdain to know the little matter that is within their power to know ; whilst they are continually aspiring at things that are without the reach of their knowledge ; things for which they have no *faculties* suited to understand, and no *data* to determine. This is not to philosophize, but to trifle.

As this Gentleman takes the liberty to sneer at the *English*, he may please to stay and take this observation along with him,---that, if it had not been for an *Englishman*, he (and such like) had known nothing. To him they are beholden for all these great and wonderful discoveries which all the world acknowledge and praise him for, except these sons of detraction, envy, and ingratitude ; discoveries that may be looked upon as the works of a genius rather divine than human.

The *English* have found, by experience, that, by the power of some cause or other, all bodies are drawn or impelled towards one another according to certain laws ; and these laws are found out by observations, and a just reasoning from the laws of motion. This cause, be it what it will, they call *gravity*, without pretending to determine of what kind it is. If this author knew the physical cause of gravity, ever so well, he would be no wiser ; for the effects of it would be just the same ; and we can measure the effects of gravity, and from thence find the quantities and proportions of the generating forces, without knowing what these forces consist in ; so that the knowledge of such a cause would be only a useless speculation. But this author is so fearful of occult causes, that he dare not make use of their manifest effects ; and condemns the English at once for believing their own eyes that there are such effects flowing from this unknown cause, which they have agreed to call *gravity* ; for it appears to them that the effects thereof are all that they have any thing to do with. Sir Isaac New-

ton tells him, more than once, *that he does not take upon him to define the kind or manner of action, or the causes or physical reason thereof*; but this author cannot take it in.

Some persons, when they can find nothing else to say, have cavilled about the term *gravity*, and so their objections are dwindled to a dispute about words, and not things. I suppose every thing in nature, that we have occasion to make the least use of in our discourse and reasoning, be it known or unknown, ought to have a name given it to distinguish it from other things, and to convey our meaning to others; and thus the *cause* of the acceleration of bodies towards one another is expressed by the word *gravity*, and the action itself is called *gravitation*. The word *attraction* is used in the very same sense; that is, not in a *physical* but *mathematical* sense, which regards only the *quantity* of the cause. That these terms are the most proper that could be chosen, will appear from hence: the word *attraction* is taken from its most manifest quality; for a body moving towards another with an accelerated motion has all the appearance possible of being drawn towards that other by some inherent virtue in the other. The word *gravity* has evermore expressed the tendency of a heavy body to the earth; and, therefore, by parity of reason, will as properly express the tendency of the moon towards the earth, or of the earth towards the sun, or of any body towards another body.

Suppose this philosopher, or any other of that fort, was asked to calculate the times, spaces, or velocities of falling bodies (which is the most simple case that can be proposed about gravity), would not any body justly laugh at him, if he stood to demur about it, and refuse to calculate till he knew whether their falling was caused by attraction, impulsion, or the rotation of a vortex? And would not he equally deserve to be laughed at, that should hesitate to calculate the motions of the moon, or of the earth and planets, or of the tides, upon the same account, when they are all acted upon by the same unknown cause of gravity?

But, to return to the tides. This gentleman (*Euler*) tells us, that Newton's method is erroneous, by which he found

the sea to rise to the height of near two feet, by the sun's force only ; and says, that Newton found out this enormous effect by comparing the sun's force with the centrifugal force of the earth. This has been answered before ; and certainly this Gentleman knows little about the nature of forces, if he does not allow that two equal forces, of however different kinds, will always have equal effects ; and proportional forces proportional effects, especially in their nascent state ; for it is not the *kind* but the *quantity* of force that is to be regarded : therefore Newton rightly found the solar tide near two feet, and the lunar tide  $8\frac{1}{2}$  feet, agreeable to experience. But, to shew you what sort of a theory this Gentleman works by, he finds the solar tide only half a foot, and the lunar tide  $2\frac{1}{4}$  feet ; in all not three feet ; which all observations confute, and, with it, his erroneous method of computation. I have met with nobody yet but what makes it at least three times as much.

He also tells us that Newton found out the forces of the sun and moon by help of the tides ; but he has not done it accurately : and yet Newton took in every circumstance that could any way affect it, as may be seen in prop. 37, b. III.

Having had occasion to compare different kinds of forces with one another, I will venture to lay down this as a general rule,---that all forces whatever, whether attractive or impulsive, centripetal or centrifugal, or of what kind soever, if they be equal, they will produce equal effects ; and, therefore, how idle must it be for these men to wrangle about the *kind*, when the *quantity* only is concerned in the effect, and can only be of any real use to us in our calculations ! The enquiry after the *kind* and *modus* of action is a physical or rather a metaphysical speculation, the knowledge of which they can never come at.

It has likewise been objected, by some persons, that the two examples of Newton for finding the tides are ill chosen ; but, however, he had no more to choose from, and by their near agreement it shews they were well chosen. Euler tells you, that, at *Havre de Grace*, the greatest and least tides are as

17 to 11; and therefore the sun's force to the moon's will be as 17 — 11 to 17 + 11, or as 6 to 28; or, as he makes it, as 7.13 to 28, which is about as 1 to 4; a proportion not very different from Newton's. *Dan. Bernoulli* says, that, at St. Malo, the greatest height to the least is as 50 to 15, which makes the sun's force to the moon's as 35 to 65, or as 7 to 13, not so much as 1 to 2; a conclusion utterly inconsistent with all other observations; which argues that the observation has not been made with sufficient accuracy. However, this is certain,—that, if any place can be improper for such an experiment, this place is, by reason of the very extraordinary tides; for here the tide, being hurried up a long channel growing continually straighter, is forced up to an unusual height.

However, I cannot think that it signifies a great deal where or in what places these experiments are made, provided the sea be deep, and have free access and receipt to and from the place of observation; for though the tides be higher in one place than another, the sun and moon conspire alike to that: for if the water be accelerated in any degree by the moon's force, it will likewise be accelerated proportionably by the sun's force, so that the result will be nearly the same.

I cannot but be surprised, that, in so material a point as this, no person has been sent purposelly to proper places to make observations of the tides; since by this method only the forces of the sun and moon can be determined to any tolerable degree of exactness. No celestial observations can assist us in this matter: astronomy affords us no help; and there seems to be no way for us to gain this great point, but this method by the tides. The forces of the sun and moon are so very small, in respect to the force of gravity, that no common hydrostatical experiment can shew us the least effects thereof: it is only in the tides that their effects become sensible.

The forces of the sun and moon reach to the very centre of the earth, and act upon every point of the radius or column of water under them, and diminish the gravity of every particle thereof; and all these forces conspire together to raise the tide;

and therefore the radius of the earth becomes the proper scale for all these forces to act on. And we cannot see the total effect of all these forces, unless we have a depth equal to the whole radius of the earth, or an extent of sea of 90 degrees; and this effect on the tides, when at the greatest, will then only amount to a few feet.

If I was to give directions for making observations on the tides for this purpose, I would advise to choose some place near the equinoctial; as, on the coasts of *Africa* or *America*: and the place would be most advantageous where the sun and moon are in the zenith at the first observation; and in the horizon at the second, for the spring tides; and one in the zenith, and the other in the horizon, for two observations at the neap tides: and such places should be chosen where the sea is of large extent, and deep, so as to communicate freely with the place, backward and forward. Such places will be best in a calm country, and performed in calm weather; and all circumstances of weather should be alike (as near as can be) at any two correspondent times of observation---the syzygies and quadratures. These observations should be made at the syzygies and quadratures, both day and night; and also when the tides are highest and lowest, which is three days after. But, for some purposes, trial should be made every day; and the business of trying at the syzygies and quadratures should be continued for a twelvemonth: and more places than one should be tried. Perhaps some island in the middle of the sea, as *St. Helena*, may be proper; for the weather is likely to be more uniform than on the continent.--- Having thus gained a sufficient number of observations, the best may be selected, and a mean ratio found, by which this matter will be finally determined.

There are some people that object to this method of finding the sun's and moon's forces by the tides; and reckon it very precarious, and subject to many obstacles and intervening causes, by which the tides are perpetually influenced and disturbed; as if every thing had not its difficulties. The only disturbing cause is the wind: yet they can tell us of no other method, but what is more precarious, more impracticable, and less exact. So much of the tides.

The 39th prop. is about finding the precession of the equinoxes: against this, being a problem of great difficulty, objections have been raised by several people; alleging, that it is not truly demonstrated. Mr. *Simpson* is among this clan; and he absurdly makes the precession by the sun's force alone to be  $21'' 6''$ , which is double to Sir Isaac Newton's: the consequence of which is, that the motion by the moon's force will be only  $29''$ ; so that, by this, the moon's force will be to the sun's only as  $1\frac{2}{3}$  to 1: yet he says, in another place, that the moon's force to the sun's cannot be less than  $2\frac{1}{2}$  to 1: so inconsistent and erroneous are his operations. But this has been taken notice of before in the Comment. But the moon's force must be greater than  $2\frac{1}{2}$ ; for, in all observations of the tides that have been regularly made, the moon's force is 4, or more: and it is hardly possible, in any observation of the tides made with any tolerable degree of care, to miss near a half. They that would see the precession of the equinoxes truly calculated, will find it in prob. 26, sect. 3, of my Fluxions.

Other objections, of less moment, I pass over, as their malignity and falsehood will appear to every reader; such as the absurd opinion that motion cannot be lost in the world (concerning which, see my Mechanics, 4to., prop. 10, and cors.); also the false opinions of those that deny that the composition and resolution of forces are analogous to the composition and resolution of motions, which are their adequate effects; their endless cavils against his method of demonstrating any proposition, which is the synthetical, preferring their own analytical methods, as being shorter, which is no wonder; their introducing into physical calculations an obscure, precarious law or force, called *vis viva*, founded upon no certain principles (concerning which, see my Mechanics, 4to., schol. to prop. 11, where the nature of it is unravelled); their promiscuously using the words force and motion for one another, which are as different as cause and effect, which induces no small obscurity into their writings; their ascribing the invention of *Fluxions* to *Leibnitz*, contrary to the clearest testimonies: and some of them tell us, that, *not he that first found, but he that first published it deserves the praise*; as if

the publisher could have published it all, if the inventor had not first found it out. Here the inventor is robbed of his due praise, to resign it to the thief that stole it: and, in general, their aggravating every trifling slip as a capital crime; and, instead of praising him for what he has done (which is more than all the world ever did before), they dispraise him for what he *has not* done, or had not time to do; and lampoon him, because his time or his knowledge was *not infinite*.

But if any body should ask what any of these *Bravos*, have done since? the answer is this----Nothing at all, or less than nothing: they have been turning science backwards; for they have been doing nothing but undermining his principles (though built upon the surest foundations), to introduce their own chimerical hypotheses, that have nothing to support them but impudence, ignorance, and presumption!

At the end of the Principia, Sir Isaac Newton has given us his thoughts of the Deity. Here he shews that God is an *eternal, infinite, and powerful Being*; that all the frame of nature is owing to him, which he made, and governs: and, from the similitude of all the parts of the world, he shews that God is *one*; that he is a Being acting with *council* and *design*, and with the greatest *wisdom*; that we have ideas only of his *attributes*, but know nothing of his *substance*, nor after what manner he acts or does any thing. Indeed, we know nothing of the substance of any thing, much less of God; and as we cannot conceive of space and time, but as necessarily existing, much more must we allow that God exists necessarily; and, consequently, that he exists *always*, and *every where*, and that not *partially*, but *totally*. In short, he has laid down the best metaphysical notions of God that can be met with any where.

But M. *Leibnitz* will not allow that God has in himself (much less other animals) any principle or power of acting, but as he is first acted on by some *motive*, which he thinks is to determine his actions; and, in consequence of that, if two equal ways of acting were laid before God, he could choose neither, but be like a balance acted on by two equal weights, that would turn no way. But at this rate God is not a *free agent*, but a mere *patient*. He reckons it an im-

*perfection* in God to be able to choose *one* out of two perfectly equal things; because, he says, there wants a *sufficient reason*, and therefore he can choose neither. But it is certainly a *greater imperfection* to choose *neither* than to choose *one*. It is a principle with him that God must have a sufficient reason; which is true: but it is God's mere will that is the sufficient reason. And how comes he to know what is a sufficient reason to a perfectly free agent that has it in his power to do any thing; as if man, who is no more than a *worm* in the creation, can presume to know what is fittest to be done?

Sir Isaac Newton has asserted, that all the great bodies in the world move in free space, which is unbounded and infinite; that all bodies have pores or empty spaces within them. But this author will not allow any *vacuum*, but will have the world be a perfect *plenum*; and he tells us, also, that the world will continue for ever without any alteration (or mending): but he has taken particular care to prevent that by introducing his *plenum*. The motions of the heavenly bodies must needs be retarded, and soon stopt, in moving through dense matter, though ever so fluid; so that one of his suppositions is inconsistent with another, and nobody could more effectually destroy his own hypothesis than he has done himself. He had had infinitely better chance for this supposition if he had made the planets, &c. to move in a vacuum.

If God had designed every thing to stagnate, and be fixed in the world, then a plenum seems to be the best constitution for that end; but as all the operations of nature are to be performed by motion, which, indeed, is the beauty of nature, then placing them in *vacuo* must needs be the best constitution, as the motions will then be free and unresisted, and the most durable possible; and therefore it is as necessary to have a *vacuum* in which these motions are to be performed, as it is necessary to have *body* to perform these motions.

The same objections he has made against empty space he makes against time; for with him space is no more than the order of co-existing beings, and time the order of successive ones; but at this rate the words *nearer* or *farther*, *sooner* or *later*, signify nothing; yet both space and time are measured by quantity, and therefore are themselves quantities, and conse-

quently real beings, which confutes his notions. And, indeed, common sense is enough to determine this; for their existence is self-evident, even to the most ignorant, and cannot be made more evident by all the arguments in the world. Arguments in such cases serve for nothing but perplexing things that are plain of themselves, and are often brought for that purpose. *Actions* and a *vacuum* are the first principles of the most antient philosophy; which is the same thing as to say, that God has made *bodies*, and *room* for them to act and move in; for the whole world appears to our senses to be nothing but *matter* and *motion*.

This learned philosopher's argument against empty space is this: *every perfection which God could give to things, without derogating from their other perfections, he has given them. Suppose, then, an empty space, God could have placed matter in it (which is more excellent than empty space); therefore he has done it, and consequently there is no vacuum.* Here he supposes that there is no excellency or perfection at all in motion; and so this world-maker will not agree to leave any room or free space for bodies to move in; and in this he is as dogmatical as if he had been originally one of God Almighty's privy council. But, had he gone one step farther, he would have seen the absurdity of it; for God did not make the world to stand still, but to move after various laws and rules; and, consequently, when this acute philosopher fills the planetary regions with matter, he *derogates from the perfections* of the other bodies by destroying their motion with his new matter; and in this case, and for this very purpose, empty space is more excellent than matter.

But the *ignis fatuus* that leads him astray is the argument of *a sufficient reason*, when, at the same time, he is no judge at all what is sufficient. Yet this principle he brandishes about with no small ostentation and assurance: but it is of no more use in his hands than the sword of *Achilles* in the hands of an infant.

*End of the Defence.*

THE  
LAWS  
OF  
*THE MOON'S MOTION*  
according to  
GRAVITY.

---

BY JOHN MACHIN.



THE  
LAWS  
OF  
THE MOON's MOTION.

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IN justice to the editor of this translation of Sir Isaac Newton's *Principia*, it is proper to acquaint the reader, that it was with my consent he published an advertisement, at the end of a volume of miscellanies, concerning a small tract which I intended to add to his book by way of appendix ; my design in which was to deliver some general elementary propositions, serving, as I thought, to explain and demonstrate the truth of the rules in Sir Isaac Newton's Theory of the Moon.

The occasion of the undertaking was merely accidental ; for he shewing me a paper which I communicated to the author, in the year 1717, relating to the motion of the nodes of the moon's orbit, I recollect that the method made use of in settling the equation for that motion was equally applicable to any other motion of revolution. And therefore I thought that it would not be unacceptable to a reader of the *Principia* to see the uses of the said method explained in the other equations of the moon's motion ; especially since the greatest part of the Theory of the Moon is laid down without any proof ; and since those propositions relating to the moon's motion, which are demonstrated in the *Principia*, do generally depend upon calculations very intricate and abstruse, the truth of which is not easily examined, even by those that are most skilful ; and which, however, might be easily deduced from other principles.

But in my progress in this design, happening to find several general propositions relating to the moon's motions which serve to determine many things which have hitherto been taken from the observations of astronomers; and having reason to think that the theory of the moon might by these means be made more perfect and complete than it is at present, I retarded the publication of the book, till I could procure due satisfaction by examining observations on places of the moon. But, finding this to be a work requiring a considerable time, not only in procuring such places as are proper, but also in performing calculations, upon a new method, not yet accommodated to practice by convenient rules, or assisted by tables, I thought it, therefore, more convenient for the bookseller not to stop the publication of his impression any longer upon this account. But that I may, in some measure, satisfy those who are well conversant in Sir *Isaac Newton's Principia* (and I could wish that none but such would look over these papers) that the said advertisement was not without some foundation, and that I may remove any suspicion that the design is entirely laid aside, I have put together, although in no order, as being done upon a sudden resolution, some of the propositions, among many others, that I have by me, which seem chiefly to be wanting in a theory of the moon, as it is a speculation founded on a physical cause; and those are what relate to the stating of the mean motions. For although it be of little or no use in astronomy to know the rules for ascertaining the mean motions of the node or apogee, since the fact is all that is wanting, and that is otherwise known by comparing the observations of former ages with those of the present, yet in matter of speculation this is the chief and most necessary thing required; since there is no other way to know that the cause is rightly assigned but by shewing that the motions are so much and no more than what they ought to be.

But that it may not be altogether without its use, I have added all the rules for the equation of the moon's motion, except two; one of which is a monthly equation of the variation

depending on the moon's anomaly; and the other an equation arising from the earth's being not in the focus of the moon's orbit, as it has been supposed to be, in all the modern theories since *Horror*.

For not having had time to examine over the observations which are necessary, but being obliged, instead thereof, to take Sir *Isaac Newton's* theory for my chief guide and direction, I cannot venture to depart from it too far in establishing equations entirely new; since I am well assured, upon the best authority, that it is never found to err more than seven or eight minutes.

And, therefore, hoping that the reader, who considers the sudden occasion and necessity of my publishing these propositions at this time, will make due allowance for the want of order and method, and look upon them only as so many distinct rules and propositions not connected, I shall begin, without any other preface, with shewing the origin of that inequality which is called the Variation or Reflection of the Moon.

The variation or reflection is that monthly inequality in the moon's motion, wherein it more manifestly differs from the laws of the motion of a planet in an elliptic orbit. *Tycho Brahe* makes this inequality to arise from a kind of libratory motion backwards and forwards, whereby the moon is accelerated and retarded by turns, moving swifter in the first and third quarter, and slower in the second and fourth, which inequality is principally observed in the octants.

Sir *Isaac Newton* accounts for the variation from the different force of gravity of the moon and earth to the sun, arising from the different distances of the moon in its several aspects.

The mean gravity of the moon to the sun, he supposes, is satisfied by the annual motion of the moon round the sun; the gravity of the moon to the earth, he supposes, is satisfied by a revolution of the moon about the earth. But the difference of the moon's gravity to the sun, more or less than the earth's gravity, he supposes, produces two effects; for, as this

difference of force may be resolved into two forces, one acting in the way, or contrary to the way, of the moon about the earth, and the other acting in the line to or from the earth, the first causes the moon to describe a larger or smaller area in the same time about the earth, according as it tends to accelerate or retard it; the other changes the form of the lunar orbit from what it ought to be merely from the moon's gravity to the earth; and both together make up that inequality which is called the variation.

But since the real motion of the moon, though a simple motion caused by a continual deflection from a straight line by the joint force of its gravity to the sun and earth, thereby describing an orbit which incloses not the earth but the sun, is yet considered as a compound motion made from two motions, one about the sun, and the other about the earth, because two such motions are requisite to answer the two forces of its gravity, if separately considered; for the very same reason, the moon's motion ought to be resolved into a third motion of revolution, since there remains a third force to be satisfied, and that is the force arising from the alteration of the moon's gravity to the sun. And this, when considered, will require a motion in a small ellipsis in the manner here described.

The circle ADFH (Fig. 1) represents the orbit of the moon about the earth in the centre T, as it would be at a mean distance, supposing the moon had no gravity to any other body but the earth. The diameter ATF divides that part of the orbit which is towards the sun, suppose ADF, from the part opposite to the sun, suppose AHF. The diameter at right angles HTD is the line of the moon's conjunction with or opposition to the sun. The figure PQLK is an ellipsis, whose centre is carried round the earth in the orbit ABDEFH, having its longer axis PL in length double of the shorter axis QK, and lying always parallel to TD, the line joining the centres of the earth and sun. Whilst the said figure is carried from A to B, the moon revolves the contrary way from Q to N, so as to describe equal areas in equal times about the centre of it; and to perform its revolution in the same time

as the centre of the said elliptic epicycle (if it may be so called) performs its revolution; the moon being always in the remoter extremity of its shorter axis in Q and K when it is in the quarters, and in the nearest extremity of its longer axis at the time of the new and full moon.

The shorter semi-axis of this ellipsis AQ is to the distance of its centre from the earth AT in the duplicate proportion of the moon's periodical time about the earth to the sun's periodical time: which proportion, if there be 2139 revolutions of the moon to the stars in 160 sidereal years, is that of 47 to 8400.

The figure which is described by this compound motion of the moon in the elliptic epicycle, whilst the centre of it is carried round the earth, very nearly represents the form of the lunar orbit; supposing it without eccentricity, and that the plane was coincident with the plane of the ecliptic, and that the sun continued in the same place during the whole revolution of the moon about the earth.

From the above construction it appears that the proportion between the mean distance of the moon and its greatest or least distances is easily assigned, being something larger than that which is assigned by Sir Isaac Newton in the 28th proposition of his third book. But as the computation there given depends upon the solution of a biquadratic equation, affected with numeral co-efficients, which renders it impossible to compare the proportions with each other so as to see their agreement or disagreement, except in a particular application to numbers, I shall therefore set down a rule, in general terms, derived from his method, which will be exact enough, unless the periods of the sun and moon should be much nearer equal than they are. Let L be the periodical time of the moon, S the period of the sun, M the synodical period of the moon to the sun, and D be the difference of the periods of the sun and moon; then, according to Sir Isaac Newton's method, the difference of the two axes of the moon's elliptic orbit, as it is contracted by the action of the sun, is to the sum of the said axes as  $3L \times \frac{M + L}{2}$  to

4DD — SS. But, according to the construction before laid down, the said proportion is as 3LL to 2SS — LL.

By Sir Isaac Newton's rule, the difference will be to the sum nearly as 5 to 694; and consequently the diameters will be nearly as 689 to 699, or 69 to 70: but, by the latter rule, the difference will be to the sum nearly as 1 to 119; and the diameters or distances of the moon, in its conjunction and quadrature with the sun, will be as 59 to 60. Dr. Halley (who in his remarks upon the lunar theory, at the end of his catalogue of the southern stars, first took notice of this contraction of the lunar orbit in the syzygies from the phenomena of the moon's motion) makes the difference of the diameters to the sum as 1 to 90; and consequently the greater axis to the lesser as  $45\frac{1}{2}$  to  $44\frac{1}{2}$ .

But the difference in these proportions of the extreme distances, though it may appear considerable, is not, however, to be distinguished by the observations on the diameters of the moon, whilst the variations of the diameters, from this cause, are intermixed with the other much greater variations arising from the eccentricity of the orbit.

The angle of the moon's elongation from the centre (Fig. 1), designed by BTN, is properly the variation or reflection of the moon; the properties of which are evident from the description.

*First,* It is as the sine of the double distance of the moon from the quadrature or conjunction with the sun; for it is the difference of the two angles BTA and NTA, whose tangents, by the construction, are in a given proportion.

*Secondly,* The variation is, *ceteris paribus*, in the duplicate proportion of the synodical time of the moon's revolution to the sun; for the variation is in proportion to the mean diameter of the epicycle, and that is in the duplicate proportion of the synodical time of revolution.

The greatest variation is an angle whose sine is to the radius as the difference of the greatest and least distances TQ and TL, that is, 3AQ, to their sum. According to the proportion of the lines before described, this rule makes the elongation near 29 minutes; which would be the variation, sup-

posing the moon performed its revolution to the sun in the time of its revolution round the earth ; but if that elongation of 29 minutes be increased in the duplicate proportion of the synodical time to the periodical time of revolution, it will produce near 34 minutes for the variation.

It is to be noted, that what is said of the epicycle, is upon supposition that the earth's orbit round the sun is a circle : if the eccentricity of the annual orbit be considered, the mean diameter of the epicycle must increase or diminish reciprocally in the triplicate proportion of the sun's distance.

The construction which I communicated to Sir Isaac Newton for the annual motion of the nodes of the moon's orbit (which is

The method of finding the inequalities in any revolution.

printed in the scholium to the 33d proposition of his third book) is a case of a general method for shewing the inequality of any motion round a centre, when the hourly motion or velocity of the object varies according to any rule, depending on its aspect to some other object ; for in any revolution the mean motion and inequality are to be assigned by means of a curvilinear figure, wherein equal areas are described about the centre in equal times ; the property of which figure is, that the rays from the centre are always reciprocally in the subduplicate proportion of the hourly motion or velocity about the centre.

Thus in the figure described in my construction (Fig. 2), where TN is the line of the nodes, TA the line drawn to the sun, is supposed to revolve round the centre T, with the velocity of the sun's motion from the node ; and the ray TB which is taken always in the subduplicate proportion of the velocity, will describe equal areas in equal times ; so that the sector NTB will be the mean motion of the sun, the sector NTA the motion of the sun from the node, and consequently the area NAB the motion of the node, which will be a retrograde motion if the area be within the circle, and direct if it falls without. From whence it follows,

1. That the periodical time of the sun's revolution to the node will be to the periodical time of the sun's revolution as the area of the curvilinear figure to the area of the circle.

2 That if a circle be described, whose area is equal to the area of the curvilinear figure, it will cut that figure in the place where the sun has the mean motion from the node.

3. If an angle NTF be made, which shall comprehend an area in the said circle, equal to the sector NTB in the figure, that angle will be the mean motion of the sun from the node. And, consequently,

4. The angle FTB, which is the difference between the sun's true motion from the node, designed by ATN, and the sun's mean motion from the node, designed by FTN, will be the equation for the sun's motion from the node, when the sun's position to the node is designed by the angle ATN.

From all which it appears, that what is said of the sun's motion from the node will hold as to any other motion round a centre; as of the sun from the moon, or the moon from the node or apogee. In any such revolution a curvilinear figure may be described about the centre, by the areas of which the relation between the mean and true motion may be shewn; and, consequently, the inequality or equation of the motion.

And as in every revolution there is a certain figure which is proper to shew this relation, such a figure may be called an equant for that motion or revolution.

And in every revolution where the equant is a figure of the same property, the inequalities or equations will alter according to the same rule.

Thus, if the equant be an ellipsis about the centre, as in that for the motion of the sun from the node,

*First*, The mean motion in the whole revolution will be a geometrical mean proportional between the greatest motion in the extremity of the lesser axis and the least motion in the extremity of the longer axis; for the radius of the circle, whch is equal to an ellipsis, is a mean proportional between the two semi-axes.

*Secondly*, The tangents of the angles of the mean and true motn are in the given proportion of the two axes of the ellipsis. Thus the tangents of the angles of the true and mean motion of the sun from the node, viz. the tangents of the

angles ATN and FTN (Fig. 2), are in proportion as the ordinates BG and FG, that is, as the semi-axes TH and TN.

*Thirdly,* The sine of the angle of the greatest inequality in the octants is to the radius as half the sum of the axes to half their difference.

It is to be noted, that the equant is an ellipsis about the centre in every motion where the excess of the velocity about the centre above the least velocity is always in the duplicate proportion of the sine of the angle of the true motion from the place where the velocity about the centre is least. From which remark, upon examination, it will appear, that the following motions are to be reduced to an elliptic equant described about the centre.

The monthly motion of the moon from the node;

The annual motion of the sun from the node;

The motion of the moon from the sun as it is accelerated or retarded by the alteration of the area described about the earth, according to Sir Isaac Newton's 26th prop. 3d book;

And the annual motion of the sun from the apogee. How these several equants are determined will appear by what follows.

The node is in its swiftest retrograde motion when the sun and moon are in conjunction or opposition, and in a quadrature with the line of the nodes. According to Sir Isaac Newton's method (explained at the end of the thirtieth proposition of the third book), the force of the sun to produce a motion in the node at this time is equal to three times the mean solar force; that is, by the construction of the elliptic epicycle, equal to a force which is to the force of gravity as 3AQ to AT (Fig. 1), or three times the lesser semi-axis of the ellipsis to the distance of its centre from the centre of the earth. But if the moon revolve in the elliptic epicycle as before described, the force to make a motion in the node at the time mentioned will be to the force of gravity as 3DL to DT, or three times the longer semi-axis to the distance of the centre; which is the double of the former force. But, then, according to Sir Isaac's method, the motion of the node at this time is to the moon's motion as the

solar force to create a motion in the node is to the force of gravity. But if the moon be conceived as revolving in a circle, with the velocity of its motion from the node at this time, when the node moves swiftest, and the plane of the said circle be supposed to have a rotation upon an axis perpendicular to the plane of the ecliptic, and the contrary way to the motion of the moon, so as to produce the motion of the node, and leave the moon to move with its own motion about the earth, the force to make a motion in the node seems to be the difference of the forces to retain it with the velocity of its motion in the moveable and immoveable planes; but the velocities of bodies revolving in circles are in the subduplicate proportion of the central forces. From whence it follows, that

*The motion of the moon from the node at this time, when the node moves swiftest, is to the motion of the moon in the subduplicate proportion of the sum of the forces to the force of gravity, or as the sum of TD and 3DL to TD.*

And this would be the greatest motion of the node, upon supposition that the plane of the moon's orbit was almost coincident with the plane of the ecliptic; but if the inclination be considered, the motive force for the node must be diminished in the proportion of the fine-complement of the inclination to the radius. How much this motion is, will appear by the following short calculation.

The distance TD being, as before, equal to 8400, and 3DL, being 282, the inclination of the plane in this position is  $4^{\circ} 59' 35''$ , the fine-complement of which is to the radius as 525 to 527 nearly; therefore the force of gravity is to the motive force for the node thus diminished in the compound proportion of 8400 to 282, and of 527 to 525, that is, in the proportion of 4216 to 141. So that the greatest motion of the moon from the node is to the motion of the moon in the subduplicate proportion of 4357 to 4216, that is, in the proportion nearly of 613 to 603. According to which calculation the greatest hourly motion of the node ought to be  $32'' 47'''$ . By Sir Isaac Newton's method, it amounts to  $33'' 10'' \frac{1}{2}$ .

This is the swiftest retrograde motion of the node, when the line of the nodes is in a quadrature with the sun, and the

moon is in its greatest latitude in conjunction or opposition to the sun. But the equant for the motion of the moon from the node in this month, when the line of the nodes is in quadrature with the sun, is an ellipsis about the centre; and therefore the mean motion in this month will be known by the following rule:

*The mean motion of the moon from the node, in that month when the line of the nodes is in a quadrature with the sun, is a geometrical mean proportional, between the greatest motion of the moon from the node and the motion of the moon.*

And therefore this mean motion will be to the motion of the moon in the subduplicate proportion of 613 to 603, that is, nearly in the proportion of 1221 to 1211. So that the mean motion of the node in this month will be to the motion of the moon as 10 to 1211, which makes the mean hourly motion  $16'' 19'' \frac{1}{40}$ . According to Sir Isaac Newton, it amounts to  $16'' 35''$ ; but, by the corrections which he afterwards uses, it is reduced to  $16'' 16'' \frac{2}{3}$ .

But the equant for the annual motion of the sun from the node being also an ellipsis, it follows, that

*The mean motion of the sun from the node is a geometrical mean proportional between the motion of the sun and the mean motion of the sun from the node in the month when the line of the nodes is in quadrature with the sun.*

How near this rule agrees with the observations, will appear by this calculation.

Since the mean motion of the node in that month when the line of nodes is in quadrature to the sun was before shewn to be to the moon's mean motion as 10 to 1211, and the motion of the sun is to the motion of the moon as 160 to 2139, it follows, that the motion of the node and the motion of the sun will be in the proportion of 154 and 1395; and, therefore, by the rule, the sun's mean motion from the node is to the sun's mean motion in the subduplicate proportion of 1549 to 1395, that is, nearly as 98 to 93; which corresponds with the observations, there being 98 revolutions of the sun to the node in 93 revolutions of the sun. The subduplicate proportion taken more nearly is as 941 to

893, which will produce  $19^{\circ} 21' 3''$  for the motion of the node from the fixed stars in a sidereal year. The motion (as observed) is  $19^{\circ} 21' 22''$ .

Had the calculation from the rule been more exactly made in large numbers, the annual motion produced would be  $19^{\circ} 21' 07\frac{1}{2}''$ , which is  $14''$  less than the motion, as observed by the astronomers.

Which difference may very probably arise from the sun's parallax; and, if so, it may perhaps furnish the best and most certain method of adjusting and fixing the true distance of the sun. For the sun's force being something more on that half of the orb which is towards the sun than what it is on the other half, the elliptic epicycle is accordingly larger in the first case than in the latter. And by calculation I find that the mean motion of the node, arising after consideration is had of this difference, is more than the mean motion from the mean magnitude of the epicycle by near  $2''$  in the year, for every minute in the parallactic angle of the orbit of the moon, or for every second of the sun's parallax. And, by the best computation I have yet made, this difference of  $14''$  in the annual motion of the node will arise from about  $8''$  of parallax; which will make the sun's distance above 25000 semi-diameters of the earth.

In like manner as the equant for the motion of the node in that month when the line of the nodes is in quadrature with the sun is an ellipsis, so in any other month it is also an ellipsis; the motion of the node being direct and retrograde by turns, in the moon's passing from the quadrature to the sun to the place of its node, and from the place of its node to the quadrature.

But these elliptic equants do not only serve to shew the inequality of the motion of the node, but also the inclination of the plane of the moon's orbit to the plane of the ecliptic. Thus the rays in the elliptic equants, for the motion of the moon from the node in each month, design the inclinations of the plane of its orbit to the plane of the ecliptic in the several respective positions of the moon to the line of the nodes. And the rays of the elliptic equant for the annual motion of

The inclination  
of the plane of the  
moon's orbit to the  
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tic.

the sun from the node, in my construction (in the schol. to prop. 33, book 3, of Sir Isaac Newton's *Principia*), design the different mean inclinations of the said plane to the plane of the ecliptic in each month, when the sun is in each respective aspect to the line of the nodes.

Thus if NT (Fig. 2) (the semi-transverse axis of the elliptic equant for the motion of the sun from the node) design the mean inclination of the plane, or, which is the same thing, if it represent the mean distance between the pole of the ecliptic and the pole of the moon's orbit, in that month when the sun is in the line of the nodes, TH, the semi-conjugate axis of the said ellipsis, will design the mean inclination, or mean distance of the poles in that month when the line of nodes is in quadrature to the sun; and TB, any other semi-diameter of the said ellipsis, will represent the mean distance between the said poles, when the sun is in that aspect to the line of nodes which is designed by the angle NTA. For example, if the least inclination designed by the short semi-axis TH be  $5^{\circ} 00' 00''$ ; since TH is to TK as the motion of the sun to the mean motion of the sun from the node, by the property of this equant; and since there are 98 revolutions of the sun to the node in 93 revolutions of the sun; it follows, that HK, the difference between the greatest and least of the mean inclinations in the several months of the year, is to TH, the least, as 5 to 93; by which proportion the said difference will amount to  $16' 10''$ . According to Sir Isaac Newton's computation, in the 35th prop. of the third book, it is  $16' 23 \frac{1}{2}''$ . But if the said number be lessened in the proportion of 69 to 70, according to the author's note at the end of the 34th prop. the said difference will become  $16' 9''$ .

And in like manner the inclinations of the plane of the moon's orbit, in that month when the motion of the node is swiftest (being situated in the line of quadratures with the sun), are determined by the equant for the motion of the moon from the node in that month.

Thus, let TH be to TN (Fig. 2) in the subduplicate proportion of the moon's motion to its greatest motion from the node, when the moon is in the conjunction in TH; that is (as was be-

fore determined), let TH be to TN in the proportion of 1211 to 1221; and the ellipsis described on the semi-axes TH and TN will be the equant for the motion of the moon from the node in that month. And the rays of the said equant will design the inclinations of the plane in the several aspects of the moon to the line of the nodes; that is, if TN be the inclination of the plane, or the distance of the pole of the ecliptic from the pole of the moon's orbit, when the moon is in TN, the line of the nodes, the ray TB will represent the distance of the said poles, or the inclination of the plane, in that aspect which is designed by the angle NTB.

Which being laid down, it follows that the whole variation of the inclination, in the time the moon moves from the line of the nodes to its quadrature in THK, is to the least inclination as KH to TH, that is, as 10 to 1211. Wherefore if the least inclination be  $4^\circ 59' 35''$ , the whole variation will be  $2' 29''$ . This is upon supposition that the sun continued in the same position to the line of the nodes during the time that the moon moves from the node to its quadrature. But, the sun's motion protracting the time of the moon's period to the sun in the proportion of 13 to 12, the variation must be increased in the same proportion, and will therefore be  $2' 41''$ . According to Sir Isaac Newton's computation, as delivered in the corollaries to the 34th prop. of the 3d book, for stating this greatest variation (the intermediate variations in this or any other month not being computed or shewn by any method), it amounts to  $2' 43''$ . But if the said quantity be diminished in the proportion of 70 to 69, according to his note at the end of the said proposition, it will become the same precisely as it is here derived from the equant.

The motion of the moon from the sun, as it is accelerated  
The variation or retarded by the increment of the area de-  
of the area de- scribed about the earth (according to the 26th  
scribed by the prop. of the 3d book), is also to be reduced to  
moon about the earth.  
an elliptic equant, by taking the shorter axis to the longer  
axis in the subquadruplicate proportion of the force of the  
moon's gravity to the earth to the said force added to three

times the mean solar force, that is, as TA (Fig. 1) to the first of three mean proportionals between TA and  $TA + 3AQ$ . And in the same proportion is the area described by the moon about the earth, when in quadrature with the sun, to the mean area, or as the mean area to the area described in the syzygies : so that the greatest area in the syzygies is to the least in the quadratures in the subduplicate proportion of  $TA + 3AQ$  to  $TA$ , or as  $\sqrt{8541}$  to  $\sqrt{8400}$ . This is upon supposition that the moon revolves to the sun in the same time as it revolves about the earth ; which will be found to agree very nearly with Sir Isaac Newton's computation in the before-cited proposition.

The motion of the apogee. And after the same manner an elliptic equant might be constructed, which would very nearly shew the mean motion of the apogee, according to the rules delivered by Sir Isaac Newton (in the corollaries of the 45th prop. of the first book) for stating the motion of the apogee, namely, by taking the greatest retrograde motion of the apogee from the force of the sun upon the moon in the quarters, and the greatest direct motion from the force of the sun upon the moon when in the conjunction or opposition; each according to his rule, delivered in the second corollary to the said proposition. And if an ellipsis be made whose axes are in the subduplicate proportion of the moon's motion from the apogee, when in the said swiftest direct and retrograde motions, the said ellipsis will be nearly the equant for the motion of the moon from the apogee, and will be found to be nearly of the form of that above for the increment of the area.

But the motion of the apogee, according to this method, will be found to be no more than  $1^{\circ} 37' 22''$  in the revolution of the moon from apogee to apogee, which (according to the observations) ought to be  $3^{\circ} 4' 7\frac{1}{2}''$ .

So that it seems there is more force necessary to account for the motion of the moon's apogee than what arises from the variation of the moon's gravity to the sun, in its revolution about the earth.

But if the cause of this motion be supposed to arise from the variation of the moon's gravity to the earth, as it revolves round in the elliptic epicycle, this difference of force, which is nearly double the former, will be found to be sufficient to account for the motion, but not with that exactness as ought to be expected. Neither is there any method that I have ever yet met with upon the commonly received principles which is perfectly sufficient to explain the motion of the moon's apogee.

The rules which follow concerning the motion of the apogee, and the alteration of the eccentricity, are founded upon other principles, which I may have occasion hereafter to explain, it being, as I apprehend, impossible to derive these, and many other such propositions, from the laws of centripetal forces.

Let  $TC$  (Fig. 1) (in the above construction of the lunar orbit) be the mean distance of the moon, or half the sum of its greatest and least distances, viz.  $TQ$  and  $TL$ ; and let  $CL$  be the mean semi-diameter of the elliptic epicycle, or half the sum of the semi-axes; and take a distance  $LM$ , on the other side towards the centre, equal to  $CL$ ; then

*The mean motion of the moon from its apogee is to the mean motion of the moon in the subduplicate proportion of  $TM$  to  $TC$ .*

For example; half the shorter axis or  $DC$  is  $23\frac{1}{2}$ ; therefore  $TC$  the mean distance is  $8376\frac{1}{2}$ ;  $CM$  or  $2CL$ , the sum of the semi-axes, is  $141$ ; so that  $TM$  is  $8235\frac{1}{2}$ . Wherefore the motion of the moon from the apogee is to the motion of the moon in the subduplicate proportion of  $8235\frac{1}{2}$  to  $8376\frac{1}{2}$ , or of  $16471$  to  $16753$ , that is, nearly as  $117$  to  $118$ , or more nearly as  $352$  to  $355$ ; or yet more nearly, as  $1877$  to  $1893$ ; so that there ought to be about  $16$  revolutions of the apogee in  $1893$  revolutions of the moon; which agrees to great precision with the most modern numbers of astronomy, according to which proportion the mean motion of the apogee in a sidereal year ought to be  $40^\circ 40' 40\frac{1}{2}''$ . But, by the numbers in Sir Isaac Newton's theory of the moon, the said

motion is  $40^{\circ} 40' 43''$ . According to the numbers of *Tycho Brahe*, it ought to be  $40^{\circ} 40' 47''$ .

The mean motion of the apogee being stated, I find the following rule for the alteration of the eccentricity. The variation of the eccentricity.

*The least eccentricity is to the mean eccentricity in the duplicate proportion of the sun's mean motion from the apogee of the moon's orbit to the sun's mean motion. Or in the duplicate proportion of the periodical time of the sun's revolution to the mean periodical time of its revolution to the moon's apogee.*

By the foregoing rule for the mean motion of the apogee there are 16 revolutions of the apogee in 1893 revolutions of the moon; but there being 254 revolutions of the moon in 19 revolutions of the sun, there must be about 7 revolutions of the apogee in about 62 revolutions of the sun, or rather about 20 in 177. So that the periods of the sun to the stars, and of the sun to the moon's apogee, are in proportion nearly as the numbers 157 and 177. The duplicate of which proportion is that of 107 to 136; which, according to the rule, ought to be the proportion of the least eccentricity to the mean eccentricity.

So that by this rule the mean eccentricity (or half the sum of the greatest and least) ought to be to the difference of the mean from the least (or half the difference of the greatest and the least), as 136 to 29.

How near this agrees with the observations, will appear from the numbers of Mr. *Horrox*, or Mr. *Flamsted*, and of Sir *Isaac Newton*:

The mean eccentricity, according to Mr. *Flamsted*, or Mr. *Horrox*, is 0.055236; half the difference between the greatest and least is 0.011617; which numbers are in the proportion of  $135\frac{1}{2}$  to  $28\frac{1}{2}$  nearly.

According to Sir *Isaac Newton*, the mean eccentricity is 0.05505; half the difference of the greatest and least is 0.01173; which numbers are in proportion nearly as  $135\frac{4}{7}$  to  $28\frac{4}{7}$ , each of which proportions is very near that above assigned.

But it is to be noted, that the rule which is here laid down is true only upon supposition that the eccentricity is exceedingly small. There is another rule derived from a different method, which presupposes the knowledge of the quantity of the mean eccentricity ; and which will not only determine the variation of the eccentricity according to the laws of gravity with greater exactness, but serve also to correct an hypothesis in the modern theories of the moon, in which their greatest error seems to consist ; and that is, in placing the earth, in the focus of that ellipsis which is described on the extreme diameters of the lunar orbit ; whereas it ought to be in a certain point nearer the perigee, as I may have occasion to explain more fully hereafter.

The greatest and least eccentricity being determined, the equant for the motion of the sun from the apogee is an ellipsis whose greater and lesser axes are the greatest and least eccentricities; and, therefore, by the property of such an equant as before laid down,

*The sine of the greatest equation of the apogee will be to the radius as the difference of the axes of the equant is to their sum; that is, as the difference of the greatest and least eccentricities to their sum.*

For example ; since the difference is to the sum as 29 to 136, by what was determined in the foregoing article, the greatest equation of the apogee will be about  $12^\circ 18' 40''$ . Sir Isaac Newton has determined it from the observations to be  $12^\circ 18'$ .

The greatest and least eccentricities being determined, the eccentricity and equation of the apogee, in any given aspect of the sun, are determined by the equant in the following manner.

Let TN be the greatest eccentricity, TH (Fig. 2) the least ; the ellipsis on the semi-axes TN and TH the equant for the motion of the apogee.

Then if the angle NTF be made equal to the mean distance or mean motion of the sun from the apogee, the angle NTB will be the true distance or motion of the sun from the apogee ; the difference BTF the equation of the apogee ; and the ray TB the eccentricity of the orbit, in that aspect of the

sun to the apogee designed by the angle NTB. Hence arises this rule:

*The tangent of the mean distance, viz. NTF, is to the tangent of the true distance NTB in the given proportion of the greatest eccentricity TN to the least TH, that is, as 165 to 107.*

From what has been laid down concerning the general property of an equant, that it is a curve line described about the centre, whose rays are reciprocally in the subduplicate proportion of the velocity at the centre, or the velocity of revolution, it will not be difficult to describe the proper curve for any motion that is proposed ; and where the inequality of the motion throughout the revolution is but small, there is no need of any nice or scrupulous exactness in the quadrature of the curve for shewing what the equation is. Thus all the small annual equations of the moon's motion arising from the different distances of the sun, at different times of the year, may be reduced to one rule exact enough for the purpose.

For since the sun's force to create these annual alterations is reciprocally in the triplicate proportion of the distance, the rays of the equant for such a motion will be in the sesquiplicate proportion of the distance. From whence it will not be difficult to prove, that, if the revolution of the motion to be equated were performed in the time of the sun's revolution, the equation would be to the equation of the sun's centre nearly as 3 to 2 : and so if the force decreased as any other power of the sun's distance, suppose that whose index is  $m$ , the equation would be to that of the sun's centre as  $m$  to 2. But if the motion be performed in any other period, the equation will be more or less in the proportion of the period of the revolution to the sun to the period of the revolution of the motion to be equated. Thus if it were the node or apogee of the moon's orbit, the equation is to the former as the period of the sun to the node or apogee to the period of the node or apogee. Which rule makes the greatest equation for the node about 8' 56", being a small matter less than that in Sir Isaac Newton's theory ; and the greatest equation for the

apogee about  $21' 57''$ , being something larger than that in the same theory.

The like rule will serve for the annual equation of the moon's mean motion. If, instead of the equation for the sun's centre, another small equation be taken in proportion to it as the force, by Sir Isaac Newton called the mean solar force, to the force of the moon's gravity, or as 47 to 8400, the said equation increased in the proportion of the sun's period to the mean synodical period of the moon to the sun, or of 99 to 8, will be the annual equation of the moon's mean motion. According to this, the equation, when greatest, will be  $12' 5''$ .

What is said may be sufficient for the present purpose, which is only to lay down the principal laws and rules of the several motions of the moon, according to gravity. Some other propositions, which seem no less necessary than the former for completing the theory of the moon's motion, as to its astronomical use, I reserve to another time.

But, to make some amends for the shortness and confusedness of the preceding propositions, I shall add one example to shew the use of the equant more at large, in what is commonly called the solution of the *Keplerian* problem; that being one of the things which I propos'd to explain, when the elements for the theory of the moon were advertised.

### *An Example of the Use of the Equant in finding the Equation of the Centre.*

LET the figure ADP (Fig. 3) be the orbit in which a body revolves, describing equal areas in equal times by lines drawn from a given point S; and let it be proposed to find the equant for the apparent motion of the said body about any other place within the orbit, suppose F.

Let there be a line FR indefinitely produced, which revolves with the body as it moves through the arc AR; and in the said line take a distance Fp, which shall be to FR, the

distance of the body from the given point F, in the subduplicate proportion of the perpendicular let fall upon the tangent of the orbit at R from the point S to the perpendicular on the said tangent let fall from the given point F; and the curvilinear figure, described by the point p, so taken every where, will be the equant for the motion of the body about the point F.

For since the areas described at the distances  $Fp$  and  $FR$  are in the duplicate proportion of those lines, that is, by the construction, in the proportion of the perpendiculars on the tangents let fall from S and F, the areas which the body describes in moving through the arc AR about the points S and F are in the proportion of the same perpendiculars; and therefore the area described by the revolution of the line  $Fp$  in the figure will be equal to that which is described by the revolution of the line SR in the orbit. So that the areas described in the figure will be equal in equal times, as they are in the orbit; and, consequently, the rays  $Fp$  of the figure will constantly be in the subduplicate proportion of the velocity of the motion, as it appears at the centre F, which is the property of the equant.

From which construction it will be easy to shew, that, in the case where a body describes equal areas in equal times about a fixed point, there may be a place found out within the orbit, about which the body will appear to revolve with a motion more uniform than about any other place.

Thus, suppose the orbit ADP was a figure, wherein the remotest and nearest apses A and P were diametrically opposite in a line passing through the point S, viz. the point about which the equal areas are described; then if the point F be taken at the same distance from the remotest apsis A as the point S is from the nearest apsis P, the said centre F will be the place about which the body will appear to have the most uniform motion; for in this case the point F will be in the middle of the figure LpDl, which is the equant for the motion about that point; so that the body will appear to move about the centre F as swift when it is in its slowest motion in

the remoter apsis A as it does when it is in its swiftest motion in the nearest apsis P.

For, by the construction, when the body is at A, the ray of the equant FL is a mean proportional between AF and AS; and when the body is at P, the ray of the equant Fl is a mean proportional between the two distances PS and PF, which are respectively equal to the former.

And in like manner, in an orbit of any other given form, a place may be found about which the motion is most regular.

If what has been said be applied to the case of a body revolving in an elliptic orbit, and describing equal areas in equal times about one of the foci, as is the case of a planet about the sun, and a secondary planet about the primary one, it will serve to shew the foundation of the several hypotheses and rules which have been invented by the modern astronomers for the equating of such motions, and likewise shew how far each of them are deficient or imperfect.

For if the ellipsis ADP be the orbit of a planet describing equal areas about the sun in the focus S; the other focus, suppose F, will be the place about which the motion is most regular, from what has been already said; that focus being at the same distance from the aphelion A as the sun at S is from the perihelion P. And, by the construction, each ray (Fp) of the equant will always be a mean proportional between FR and RS, the two distances of the planet from the two foci, in that place where the ray Fp is taken; for the rays SR and RF, making equal angles with the tangent at R, by the property of the ellipsis, are in the proportion of the perpendiculars from S and F, let fall on those tangents. And, therefore, Fp being to FR in the subduplicate proportion of SR to FR, it will be a mean proportional between those distances.

1. Hence when the planet is in the aphelion A, or perihelion P, the rays of the equant FL and Fl are the shortest, each being equal to CD, the lesser semi-axis of the orbit; for, by the property of the ellipsis, the rectangle of the extreme distances from the focus is equal to the square of the lesser semi-axis.

2. When the planet is at its mean distance from the sun in D or d, the extremities of the lesser axis, the equant cuts the orbit in the same place; the rays of the equant being then the longest, being each equal to the greater semi-axis CA. For in those points of the orbit the distances from the foci and the mean proportional are the same.

From which form of the equant it appears,

1. That the velocity of the revolution about the focus F diminishes in the motion of the planet from the aphelion or perihelion to the mean distance, and increases in passing from the mean distance to the perihelion or aphelion; for the rays of the equant increase in the first case, and diminish in the latter; and the velocity of revolution increases in the duplicate proportion, as the rays diminish.

2. In any place of the orbit, suppose R, the velocity of the revolution about the focus F is in proportion to the mean velocity, as the rectangle of the semi-axes of the orbit CD and CA to the rectangle of the focal distances RF and RS; for the equant and the orbit, being figures of the same area, are each equal to a circle whose radius is a mean proportional between the two semi-axes CD and CA. But the mean motion about the focus F is in those places where the said circle cuts the equant; and in other places the velocity of the revolution is reciprocally as the square of the distance, that is, reciprocally as the rectangle of the focal distances RF and RS.

3. So that the planet is in its mean velocity of revolution about the focus F in four places of the orbit, that is, where the rectangle of the focal distances is equal to the rectangle of the semi-axes; which places in orbits nearly circular, such as those of the planets, are about 45 degrees from the aphelion or perihelion; but may be assigned in general, if need be, by taking a point in the orbit, suppose R, whose nearest distance from the lesser axis of the orbit CD is to the longer semi-axis CA in the subduplicate proportion of the longer axis to the sum of the two axes; as may be easily proved.

What has been said may be enough to shew the form of the equant, and the manner of the motion about the upper

focus in general; but the precise determination of the inequality of the motion requires the knowledge of the quadrature of the several sectors of the equant, or, at least, if any other method be taken, of that which is equivalent to such a quadrature.

There are divers methods for shewing the relation between the mean and true motion of a planet round the sun, or round the other focus, some more exact than others; but the following seems the most proper for exhibiting in one view all the several hypotheses and rules which are in common use in the modern astronomy, whereby it may easily appear how far they agree or differ from each other, and how much each of them errs from the precise determination of the motion, according to the true law of an equal description of areas about the sun.

Upon the centre F describe the ellipsis LNI, equal and similar to the elliptic orbit ADP, but having its axes FN and FL contrarily posited, that is, the shorter axis LF lying in the longer axis of the orbit AP, and the longer axis FN parallel to the shorter CD. Let the focus of the said ellipsis be in f; and suppose two other ellipses LBl and Lfl to be drawn upon the common axis Ll, one passing through the point B, where the perpendicular FN intersects the orbit, and the other through the focus f. Let the line FR, revolving with the planet in the orbit, be indefinitely produced, till it intersect the first ellipsis LNI (which was similar to the orbit) in Q, the equant in p, and the ellipsis LBl (drawn through the intersection B) in K. From the point K let fall KH perpendicular to the line of apsides AP, and let it be produced till it intersect the first ellipsis LNI in O, and the ellipsis Lfl (passing through the focus f) in E; and, lastly, in the ellipsis LNI let GM be an ordinate equal and parallel to EH. In which construction it is to be noted that the ellipses Lfl and LBl are supposed as drawn only to divide the line OKH in given proportions, that KH may be to OH as the *latus rectum* of the orbit to the transverse axis; and that EH or GM, the base of the elliptic segment GLM, may be to OH as the distance of the foci to the transverse axis.

Which being premised, it will be easy to prove that the sector pFL in the equant, or, which is the same thing, the sector RSA in the orbit, is equal to the curvilinear area OKFMG, that is, equal to the elliptic sector QFL, deducting the segment LMG, and adding or subducting the trilinear space QKO, according as the angle RFA is less or greater than a right angle. Wherein it is to be noted that these signs of addition and subduction are to be used in general, if the angle AFR is taken from the aphelion in the first semi-circle, but towards the aphelion in the latter semi-circle; but if the angle AFR be taken the same way throughout the whole revolution, as is the method in astronomical calculations, then the segment and the trilinear space in the latter semi-circle must be taken with the contrary signs to what are laid down.

Hence it appears that the inequality in the motion of a planet about the upper focus F consists of three parts.

I. The first and principal of which is the inequality in the alteration of the angle QFL in making equal areas in the ellipsis LNI (Fig. 3); for if a circle equal to the ellipsis be described upon the centre F, since the radius (being a mean proportional between the two semi-axes) will fall without the ellipsis about the line of apsides, and within it about the middle distances, the angle QFL, which is proportional to the area described in the circle, will therefore increase faster about the line of apsides, and slower about the middle distances, in describing equal areas in the ellipsis, than it ought to do in the hypothesis of Bishop *Ward*, who makes the planet revolve uniformly about the focus. The equation to rectify this inequality is determined by the following rule.

The tangent of the angle QFL is to the tangent of the angle in the circle including the same area as the longer axis of the ellipsis to the shorter axis; and the difference of the angles whose tangents are in this proportion is the equation; as is manifest from what was before said on the properties of an elliptic equant. From the same it also follows, that,

1. The greatest equation is an angle whose sine is to the radius as the difference of the axes to their sum, or, which is

the same thing, as the square of the distance of the foci to the square of half the sum of the axes. So that in ellipses nearly circular, of different eccentricities, this greatest equation will vary nearly in the duplicate proportion of the eccentricity.

2. In ellipses nearly circular, the equation at any given angle QFL is to the greatest equation nearly as the sine of the double of the given angle to the radius; which follows from hence, that the equation is the difference of two angles whose tangents are in a given proportion, and nearly equal.

3. This equation adds to the mean motion in the first and third quadrant of mean anomaly, and subducts in the second and fourth; as will easily appear from that the line QF, in describing equal areas in the ellipses, makes the angle to the line of the apsides less acute than it would be in an uniform revolution.

This is the equation which is accounted for in the hypothesis of *Bullialdus*; for he supposes the motion of the planet in its orbit to be so regulated about the upper focus, that the tangents of the angles, from the lines of apsides, shall always be to the tangents of the angles answering to the mean anomaly in the proportion of the ordinates in the ellipsis to the ordinates in the circle circumscribed; which in effect is the same as if he had made the true equant for its motion about the focus F to be the ellipsis as above described.

The same equation is also used by Sir Isaac Newton in his solution of the *Keplorian* problem, in the scholium to the 31st prop. of the 1st book, and is there designed by the letter V.

But since the true equant LDI coincides with the elliptic equant in the extremities of the shorter axis at L and I, and falls within the same at its intersection with the longer axis FN, it follows, that the motion of the planet in the semi-circle about the aphelion is swifter than according to the hypothesis of an equal description of areas in the ellipsis LNI, and for the same reason slower in the other semi-circle about the perihelion; the velocity about the centre F being always reciprocally in the duplicate proportion of the distance.

Which leads to the second part of the inequality of the motion about the focus.

II. The equation to rectify this inequality is an angle answering to the segment GLM; which angle is to be added to the mean anomaly, to make the area of the elliptic sector QFL.

This angle or equation is determined by the following rule. Let R be an angle subtended by an arc equal in length to the radius of the circle, viz. 57,29578 degrees; and let A be an angle whose sine is to the radius as GM, the base of the segment, to FN, the semi-transverse axis; also let B be an arc in proportion to R, as the sine of the double of the angle A to the radius: then the equation for the segment will be equal to  $A - \frac{1}{2}B$ .

This equation is at its maximum when the angle LFQ is a right angle; the base of the segment becoming equal to Ff, half the distance of the foci; and the angle A, being in this case half the angle FDS formed at the extremity of the lesser axis, and subtended by FS, the distance of the foci; which is commonly called the greatest equation of the centre. And, consequently, the arc B, in this case, is to R as the sine of the said greatest equation of the centre is to the radius. So that, according to this rule, for the measure of the segment, it will follow, that,

1. This greatest equation is in proportion to the greatest equation of *Bullialdus*, as found in the preceding article for the elliptic equant, nearly as three times the transverse axis to eight times the distance of the foci. Or, otherwise, the greatest equation is to the angle designed by R as twice the cube of the distance between the foci to three times the cube of the transverse axis. Either of which rules may be derived from the true angle, as before determined; or by taking  $\frac{2}{3}$  of the rectangle of GM and LM, the base and height of the segment, for the measure of that segment.

So that in elliptic orbits nearly circular this greatest equation for the segment is in the triplicate proportion of the eccentricity.

2. This equation at any given angle QFL is to the greatest equation in the triplicate proportion of the ordinate OH to the semi-transverse; that is, nearly as the cube of the sine of the mean anomaly joined to the double of *Bullialdus's* equation to the cube of the radius. For the segment GML, which is proportional to the equation, is in the triplicate proportion of its base nearly; and the base is proportional to the ordinate OH, by the construction.

But the ordinate OH (in a circle described upon the radius FN) becomes the sine of an angle whose tangent is to the tangent of the angle QFL in the proportion of the transverse axis to the conjugate; but the tangent of the same angle QFL is to the tangent of the mean motion answering to the area of the elliptic equant QFL in the same proportion. So that the ordinate OH is to the sine of that angle of mean motion in the duplicate of the said proportion; and consequently the ordinate OH, in the circle on the radius FN, is the sine of an angle nearly equal to the mean anomaly joined to the double of *Bullialdus's* equation.

3. This equation adds to the mean motion in passing from the aphelion to the perihelion, and subducts in passing from the perihelion to the aphelion; as is evident from the transit of the point of intersection E round the periphery of the ellipse Lfl.

In Sir Isaac Newton's rule (in the before-cited scholium to the 31st prop. 1st book) the angle X answers to this equation for the segment; excepting that it is there taken in the triplicate proportion of the sine of the mean anomaly, instead of the triplicate proportion of the ordinate OH. The error of this rule makes,

III. The third part of the inequality, answering to the trilinear space OKQ, being the difference of the elliptic sector OFQ and the triangle OFK.

The sector OQF is proportional to an angle which is the difference of two angles whose tangents are in the given proportion of the semi-latus rectum FB and the semi-transverse FN, or in the duplicate proportion of the lesser axis to the axis of the orbit. So that this sector, when at a maximum,

is as an angle whose sine is to the radius as the difference of the *latus rectum*, and transverse to their sum; or as the difference of the squares of the semi-axes to their sum.

The triangle OFK is proportional to the rectangle of the co-ordinates OH and HF; that is, as the rectangle of the sine OH and its cosine, in the circle on the radius FN; or as the sine of the double of that angle whose sine is OH; that is, the double of the angle whose tangent is to the tangent of the angle QFL in the given ratio of the greater to the lesser axis; or whose tangent is the tangent of the angle of mean motion answering to the elliptic sector QFL in the duplicate of the said ratio. But this triangle OFK, when at a maximum, makes an angle of mean motion which is to the angle called R as BN, half the difference between the *latus rectum* and transverse axis, is to the double of the transverse axis.

So that the sector or triangle in orbits nearly circular is always nearly equal to the double of *Bullialdus's* equation.

The triangle and sector being thus determined, the equation for the trilinear space is accordingly determined. From what has been said, it appears, that,

1. This equation for the trilinear space OKQ is to that for the triangle OKF in a ratio compounded of BN, the difference between the semi-transverse and semi-latus rectum to the semi-latus rectum, and of the duplicate proportion of the sine OH to the radius; or OKQ is to OKF in a proportion compounded of the duplicate proportion of the distance of the foci to the square of the lesser axis, and the duplicate proportion of the sine OH to the radius; for the trilinear figure OKQ and the triangle OKF are nearly as OK and KH, which are in that proportion; and consequently it holds in this proportion to the double of *Bullialdus's* equation.

2. This equation, in different angles, is as the content under the sine complement and the cube of the sine; for the triangle OKF is as the rectangle of the sine and the sine complement.

3. It is at a maximum at an angle whose sine complement is to the radius as the square of the greater axis is to the

sum of the squares of the two axes; which in orbits nearly circular is about 60 degrees of mean anomaly.

4. In orbits of different eccentricities it increases in the quadruplicate proportion of the eccentricity.

5. It observes the contrary signs to that for the elliptic equant, called *Bullialdus's* equation; subduing from the mean motion in the first and third quadrants, and adding in the second and fourth, if the motion is reckoned from the aphelion.

The use of these equations in finding the place of a planet from the upper focus will appear from the following rules, which are easily proved from what has been said.

Let  $t$  be equal to CA the semi-transverse,  $c$  equal to FC the distance of the centre from the focus,  $b$  equal to CD the semi-conjugate, and  $R$  an angle subtended by an arc equal to the radius, viz.  $57^\circ 17' 44'' 48'''$ , or 57,2957795 degrees.

$$\text{Take an angle } T = \frac{cc}{2tt} R; E = \frac{b}{2t} T; S = \frac{4c}{3b} T.$$

The angle  $T$  will be the greatest equation for the triangle OFK; the angle  $S$  will be the greatest equation for the segment LMG; and the angle  $E$  will be the greatest equation for the area OKFL. Which greatest equations being found, the equations at any angle of mean anomaly will be determined by the following rules.

Let  $M$  be the mean anomaly; and let  $\tau$  be to  $T$  as the sine of the angle  $2M$  to the radius: in which proportion, as also in the following, there is no need of any great exactness, it being sufficient to take the proportions in round numbers.

Take  $e$  to  $E$  as the sine of  $2M \pm 2\tau$  to the radius; and  $s$  to  $S$  as the cube of the sine of  $M \pm \tau$  to the cube of the radius.

Then the angle QFL is equal to  $M + e + s$ , in the first quadrant LN; or  $M - e + s$ , in the second quadrant Nl; or  $M + e - s$ , in the third quadrant: or  $M - e - s$ , in the fourth quadrant.

Note, That the small equation  $\tau$  is always of the same sine with the equation  $e$ ; and in the case of the planets always near the double of that equation.

The angle RFA at the upper focus F being known, the angle RSA at the sun in the other focus is found by the common rule of Bishop *Ward*; viz. the tangent of half the angle RSA is to be to the tangent of half the angle RFA always in the given proportion of the perihelion distance SP to the aphelion distance SA. How these equations are in the several eccentricities of the moon's orbit, will appear by the following table.

| Eccentr. | E.   | S. |
|----------|------|----|
|          | "    | "  |
| 0.040    | 1.23 | 09 |
| 0.045    | 1.45 | 13 |
| 0.050    | 2.09 | 17 |
| 0.055    | 2.36 | 23 |
| 0.060    | 3.06 | 30 |
| 0.065    | 3.38 | 38 |
| 0.070    | 4.14 | 47 |

To add one example: suppose the eccentricity 0.060, the mean anomaly  $30^\circ$ . The fine of the double of the mean anomaly, that is, the fine of 60, is to the radius nearly as 87 to 100; whence, if the equation  $E = 3' 06''$  be divided in that proportion, it will produce  $2' 40''$  nearly for the equation  $e$ : the fine of  $M$  is, in this case, equal to  $\frac{1}{2}$  the radius, the cube is  $\frac{1}{8}$  of the cube of the radius; whence if the equation  $S = 30''$  be divided in the same proportion, it will produce near  $4''$  for the equation  $s$ . Therefore the angle RFA, which is  $M + e + s$ , will be  $30^\circ 2' 44''$ ; and the half is  $15^\circ 1' 22''$ : wherefore if the tangent of this angle be diminished in the proportion of 1.06, the aphelion distance, to 94, the perihelion distance, it will produce the tangent of  $13^\circ 23' 13''$ ; the double of which,  $26^\circ 46' 26''$ , is the true anomaly or angle at the sun RSA. And consequently the equation of the centre is  $3^\circ 13' 34''$  to be subducted, at 30 degrees mean anomaly.

When the place of a planet is found by this or any other method, the place may be corrected to any degree of exactness by the common property of the equant, viz. that the rays are reciprocally in the duplicate proportion of the velocity about the centre; for in this case, if there be a difference

between the mean motion belonging to the angle assumed at the upper focus and the given mean motion, the error of the angle assumed is to the difference as the rectangle of the semi-axes to the rectangle of the distances from the foci. But in orbits like those of the planets the rules as they are delivered above are sufficient of themselves, without farther correction.

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## POSTSCRIPT.

UPON reviewing these few sheets after they were printed off, which happened a little sooner than I expected, I fear the apology I have offered for delivering the propositions relating to the moon's motion in this rude manner, without giving any proof of them, or so much as mentioning the fundamental principles of their demonstration, will scarcely pass as a satisfactory one; especially since there are among these propositions some which, I am apt to think, cannot easily be proved to be either true or false by any methods which are now in common use.

Wherefore to render some satisfaction in this article, I shall add a few words concerning the principles from whence these propositions and others of the like nature are derived; and also take the opportunity to subjoin a few remarks which ought to have been made in their proper places.

*First,* There is a law of motion, which holds in the case where a body is deflected by two forces tending constantly to two fixed points;

Which is, *That the body, in such a case, will describe, by lines drawn from the two fixed points, equal solids in equal times about the line joining the said fixed points.*

The law of *Kepler*, that bodies describe equal areas in equal times about the centre of their revolution, is the only general principle in the modern doctrine of centripetal forces.

But since this law, as Sir *Isaac Newton* has proved, cannot hold, whenever a body has a gravity or force to any other than one and the same point, there seems to be wanting

Some such law as I have here laid down, that may serve to explain the motions of the moon and satellites, which have a gravity towards two different centres.

It follows, as a corollary to the law here laid down, that if a body, gravitating towards two fixed centres, be supposed, for given small intervals of time, as moving in a plane passing through one of the fixed centres, the inclination of the said plane to the line joining the centres will vary according to the area described; that is, if the area be greater, the inclination will be less; and if the area be less, the inclination will be greater, in order to make the solids equal.

This corollary, when rightly applied, will serve to explain the variation of the inclination of the plane of the moon's orbit to the plane of the ecliptic.

And how extremely difficult it is to compute the variation of the inclination in any particular case, without the knowledge of some such principle as this is, will best appear, if any one consider the intricacy of the calculations used in the corollaries to the 34th prop. of the third book of the *Principia*, in order to state the greatest quantity of variation in that month when the line of the nodes is in quadrature with the sun, and that only in particular numbers, whereby it is determined to be  $2' 43''$ .

Whereas, there is a plain and general rule in this case, which follows from what is laid down, though not immediately; namely, that the greatest variation in the said position of the moon's orbit is to the mean inclination of the plane as the difference of the greatest and least areas described in the same time by the moon about the earth, when in the conjunction and in the quarters, to the mean area.

Wherefore if  $S$  be to  $L$  as the sun's period to the moon's period, the greatest area is to the least as  $\sqrt{S}S + \frac{3}{2}LL$  to  $S$ , or as  $S + \frac{3}{2}LL$  to  $S$  nearly, by what is said on this article in the 208th page. So that the difference of areas is to the mean area as  $\frac{3}{2}LL$  to  $SS + \frac{3}{4}LL$ ; and in the same proportion is the greatest variation of the inclination of the plane

in this month to the mean inclination, which agrees nearly with Sir Isaac's computation.

Secondly, There is a general method for assigning the laws of the motion of a body to and from the centre, abstractly considered, from its motion about the centre.

The motion to and from the centre is called by Kepler a libratory motion; the knowledge of which seems absolutely requisite to define the laws of the revolution of a body, in respect of the apsides of its orbit.

For the revolution of a body, from apsis to apsis, is performed in the time of the whole libratory motion; the apsides of the orbit being the extreme points wherein the libratory motion ceases.

So that, according to this method, the motion of a body round the centre is not considered as a continued deflection from a straight line, but as a motion compounded of a circulatory motion round the centre, and a rectilinear motion to or from the centre.

Each of which motions require a proper equant. Of the equant for the motion round the centre I have already given several examples; and in the case of all motions which are governed by a gravity or force tending to a fixed point, the real orbit in which the body moves is the equant for this motion. In all other cases it is a different figure.

The equant for the libratory motion is a curve line figure, the areas of which serve to shew the time wherein the several spaces of the libration are performed.

Which figure is to be determined by knowing the law of the gravity to the centre; for the libratory force to accelerate or retard the motion to or from the centre is the difference between the gravity of the body to the centre and the centrifugal force arising from the circulatory motion. But the latter is always under one rule; for in all revolutions round a centre, in any curve line, whether described by a centripetal force or not, the centrifugal force is directly in the duplicate proportion of the area described in a given small time, and reciprocally in the triplicate proportion of the distance; which is an immediate consequence of a known

proposition of Mr. *Huygens*. The like proportion also holds as to the centripetal force in all circular motions, from a known proposition of Sir *Isaac Newton*. But what is true of the centripetal force in circles is universally true of the other force in orbits of any form.

So that by knowing the gravity of the body, since the other force is always known, the difference, which is the absolute force to move the body to or from the centre, will be known; and from thence the velocity of the motion, and the space described in any given time, may be found, and the equant described. These hints may be sufficient to shew what the method is.

To add an example. If the gravity be reciprocally as the square of the distance, the equant for the libratory motion will be found to be an ellipsis similar to the orbit, whose longer axis is the double of the eccentricity: the centre of the libratory motion, that is, the place where it is swiftest, will be in the focus; the time of the libration through the several spaces is to be measured by sectors of the said ellipsis, similar to those described by the body round the focus of the orbit; and the period of the libratory motion will be the same with the period of the revolution.

In any other law of gravity, the equant for the libratory motion will either be of a form different from the orbit, or, if it be of the same form, it must not be similarly divided.

I may just mention that the equant for the libratory motion, in the case of the moon, is a curve of the third kind, or whose equation is of four dimensions; but is to be described by an ellipsis, the centre of the libration not being in the focus.

From this method of resolving the motion it will not be difficult to shew the general causes of the alteration of the eccentricity and inequality in the motion of the apogee; for when the line of apsides is moving towards the sun, it may be easily shewn, that, since the external force in the apsides is then centrifugal, it will contribute to lengthen the space and time of the libration: by lengthening the space, it increases the eccentricity; and by lengthening the time of the

libration, it protracts the time of the revolution to the apsis, and causes what is improperly called a motion of the apsis forward. But when the line of apsides is moving to the quadratures, the external force in the apsides is at that time centripetal, which will contribute to shorten the space and time of libration; and by shortening the space will thereby lessen the eccentricity, and by shortening the time of libration will thereby contract the time of the revolution to the apsis, and cause what is improperly called a retrograde motion of the apsis.

I shall only add a few remarks, which ought to have been made in their proper places.

As to the motion of the moon in the elliptic epicycle (p. 198), it should have been mentioned, that there is no need of any accurate and perfect description of the curve called an ellipsis, it being only to shew the elongation of the moon from the centre of the epicycle, which doth not require any such accurate description.

It should have been said (Fig. 1), that when the moon is in any place of its orbit, suppose somewhere at N, in that half of the orbit which is next the sun, it then being nearer the sun than the earth, has thereby a greater gravity to the sun than the earth: which excess of gravity, according to Sir Isaac Newton's method, consists of two parts; one acting in the line NV, parallel to that which joins the earth and sun, and the other acting in the line VB directed to the earth; and these two forces, being compounded into one, make a force directed in the line NB, which is in proportion to the force of gravity as that line NB is to TB nearly. Wherefore as there is a force constantly impelling the moon somewhere towards the point B, this force is supposed to inflect the motion of the moon into a curve line about that point; for the same reason as the gravity of it to the earth is supposed to inflect its motion into a curve line about the earth: not that the moon can actually have so many distinct motions, but the one simple motion of the moon round the sun is supposed to arise from a composition of these several motions.

In the last article on the small annual equations (page 213), these rules ought to have been added.

Let  $\Delta E$  be the equation of the sun's centre;  $P$  the mean periodical time of the node or apogee;  $S$  the mean synodical time of the sun's revolution to the node or apogee: then will  

$$\frac{3S}{2P} \Delta E$$
 be the annual equation of the node or apogee, according as  $S$  and  $P$  are expounded.

The like rule will serve for the annual equation of the moon's mean motion. If  $S$  be put for the sun's period,  $P$  for the mean synodical period of the moon to the sun, and  $L$  for the moon's period to the stars, the annual equation of the moon's mean motion will be  $\frac{3LL}{2PS} \Delta E$ .

According to these rules, when expounded, the equation for the node will be found to be always in proportion to the equation of the sun's centre, nearly as 1 to 13;

The equation of the apogee to the equation of the sun's centre as 10 to 53;

And the equation of the moon's mean motion to the same as 8 to 77.

It may be throughout observed, that the propositions are in general terms, so as to serve, *mutatis mutandis*, for any other satellite as well as the moon.

There might have been several other observations and remarks made in many other places, had there been sufficient time for it; but, perhaps, what I have already said may be too much, considering the manner in which it is delivered.

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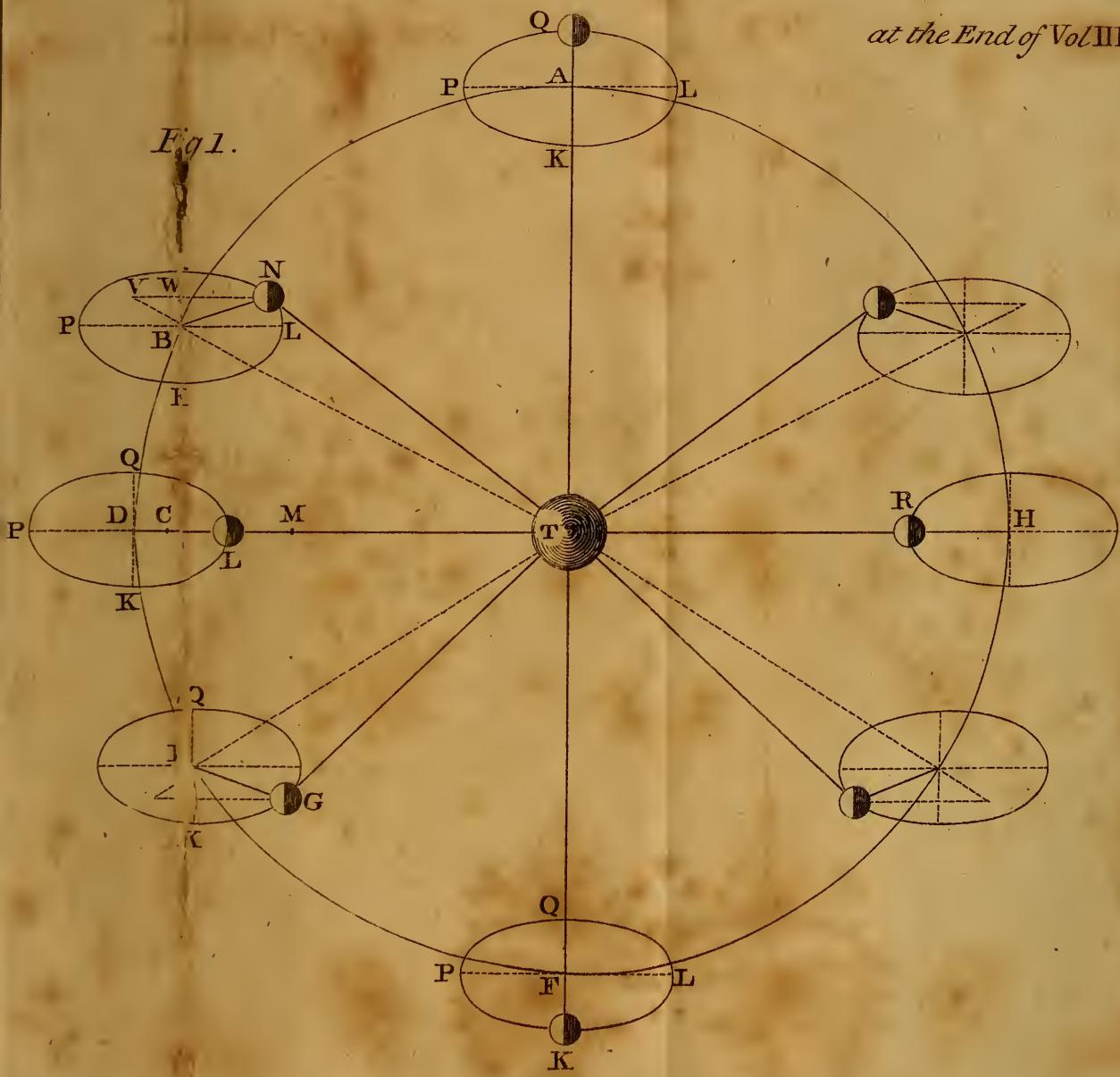
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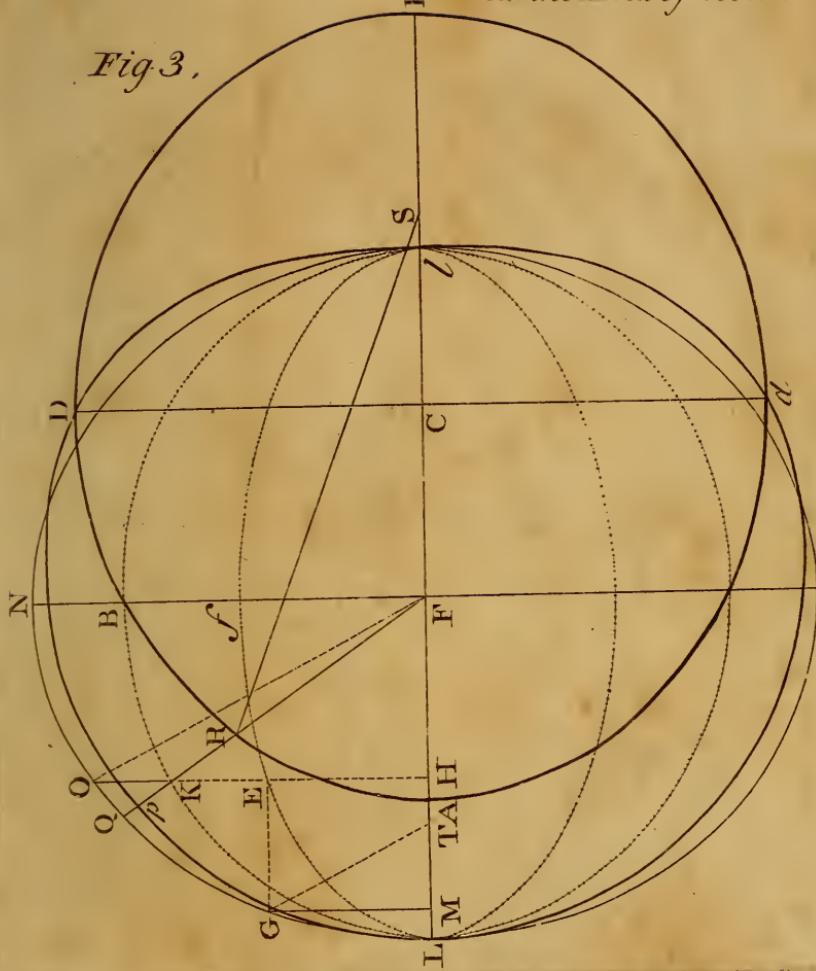
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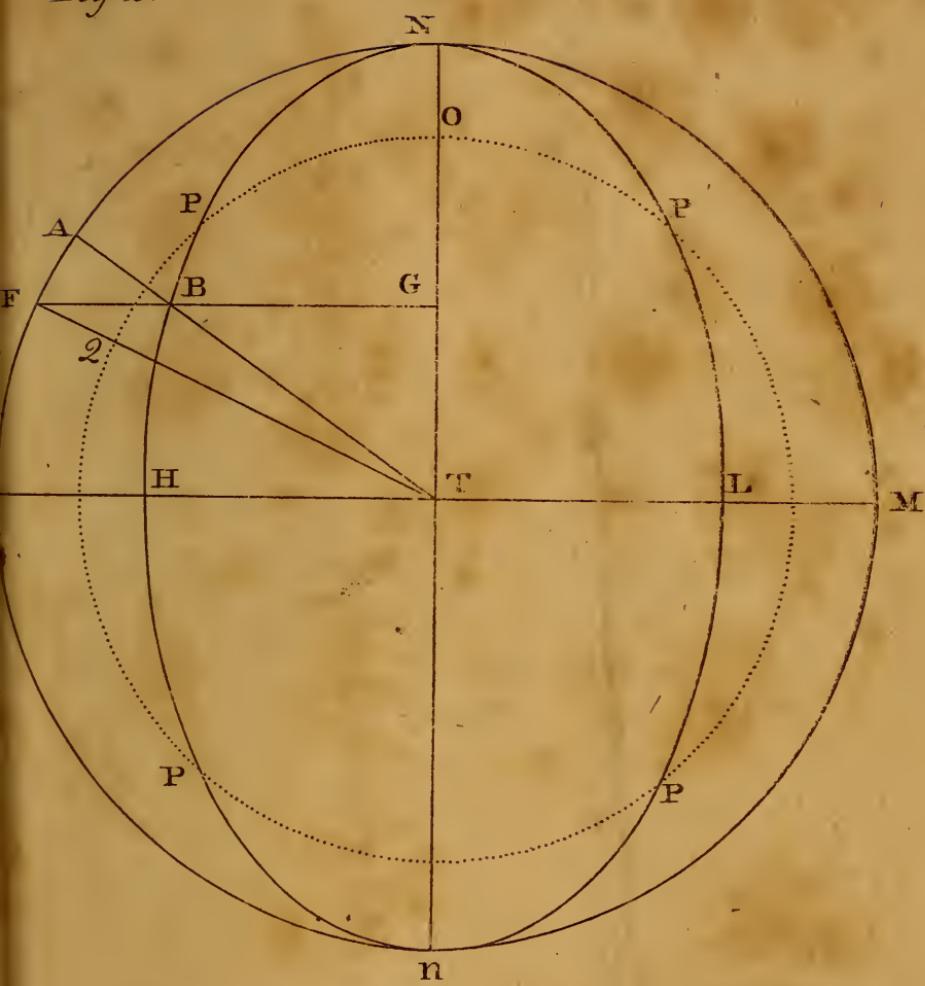
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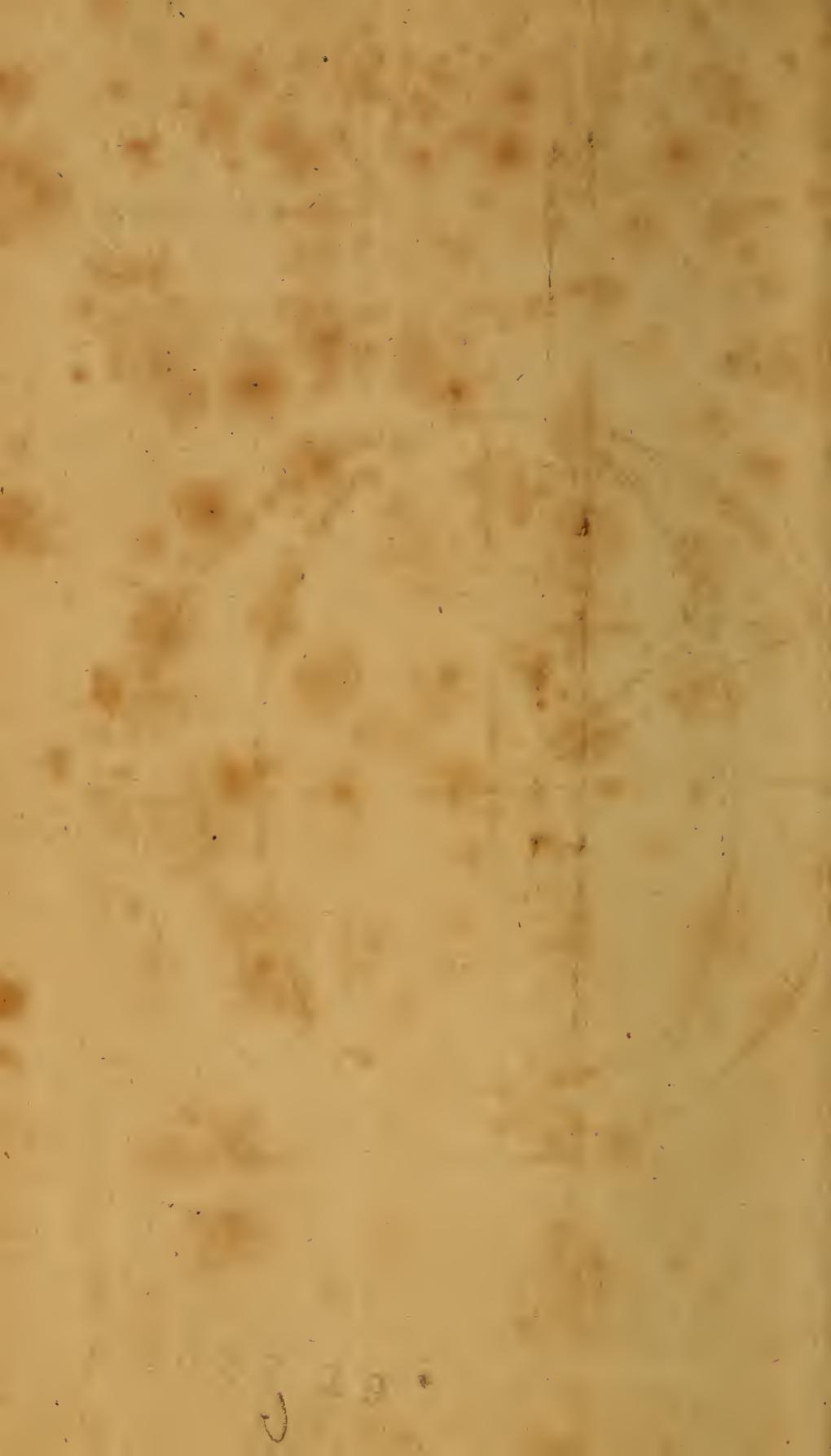




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*Fig 2.*











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